Noncommutative geometry and theoretical physics

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Abstract. *The structure of a manifold can be encoded in the commutative algebra of functions on the manifold it self – this is usual –. In the case of a non commutative algebra there is no underlying manifold and the usual concepts and tools of differential geometry (differential forms, De Rham cohomology, vector bundles, connections, elliptic operators, index theory ...) have to be generalized. This is the subject of non commutative differential geometry and is believed to be of fundamental importance in our understanding of quantum field theories. The present paper is an introduction for the non specialist and a review of the principal results on the field.*

1. INTRODUCTION

The interplay between mathematics and physics, and in particular between geometry and quantum field theory, has played an important role during the last fifteen years. Most of the tools handled by theoretical physicists involve usually an underlying smooth manifold of real dimension 3, 4 or more (the description of string theory involves loop spaces which are infinite dimensional). The geometrical description of a quantized field interacting with several external other fields has reached a satisfactory status (think, for

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example, of a Dirac operator acting on spinors coupled to an external Yang Mills field); however this is not yet the case for a fully interacting quantum field theory (even in the «simple» case of quantum electrodynamics where we have *two* coupled equations, namely the Dirac equation for ψ coupled to A_μ and the Maxwell equation for A_μ coupled to the vector field $\bar{\psi}\gamma_\mu\psi$). At the first quantized level, a field theory can usually be described in terms of sections of vector bundles and of operators (linear or not) acting on them. At the second quantized level, a quantum field theory, cannot be described in such simple terms. For instance when we consider the space Γ of sections $\varphi(x)$ of some vector bundle endowed with a real scalar product and build the infinite dimensional Clifford algebra Cliff(Γ), thus turning the classical fields $\varphi(x)$ into (second quantized) anticommuting quantum fields, we are no longer doing some «classical» geometry on a smooth manifold.

Our aim in this introduction is to explain what noncommutative geometry is about. Before that, we should remind the following result (Gelfand): all the properties of a space X can be encoded in the algebra of functions $C(X)$ on this space (1) (and conversely of course). In particular, the topology,the measure theory, (cf. Remarks in §12), the De Rham theory, the K-theory (*etc.*) of a smooth manifold of M can be described (and defined) directly in terms of the algebra $C(M)$ of functions on M . For instance the space of sections of a vector bundle over M will be defined as a module – a representation space – for the algebra $C(M)$ (actually it is a projective module of finite type cf. §9). $C(M)$ being a commutative algebra, usual geometry (and classical field theory) is, in a sense, a «commutative geometry». In order to get results in noncommutative geometry, one may proceed as follows: first choose a geometrical notion that you know how to formulate in terms of a space X , then express this notion in terms of the commutative algebra $C(X)$, finally, try to define this notion in such a way that it makes sense for an arbitrary noncommutative (but associative) algebra A . Of course, in this way, usual geometry (*i.e.* commutative geometry) will appear as a particular case of noncommutative geometry. Notice that physicists have already followed this path when they have discovered supersymmetries: indeed, superfunctions (and superfields) do not appear as functions on a usual space since they would make a commutative algebra in this case. Supergeometry (cf. [1],[2]) can be thought of as the first step beyond usual geometry, namely the passage from commutative algebra to graded commutative algebras.

(1) Let A a commutative algebra and x be an irreducible representation of A . Since A is commutative, we may choose the complex numbers as representation space where x acts by multiplication, *i.e.*, if f belongs to A , $x[f]$ is a complex number. Let us call $X(= spA)$ the space of irreducible representations of A ; then, to each f in A , we may associate a function on X , still denoted by f via the following beautifully simple relation $f(x) = x[f]$. Therefore $A = C(X)$. More precisely, one should take A as a Banach algebra with unit and talk about maximal ideals rather than irreducible representations but the idea is the same.

The purpose of noncommutative geometry is to go beyond that and to provide us with the mathematical tools required to study noncommutative algebras as noncommutative «spaces».

Usual tools of differential geometry have an analogue in a noncommutative context and their use in order to develop a theory of interacting strings has been advocated in [3]. However, noncommutative geometry has many more possible applications in quantum field theory, in statistical physics or even in solid state physics (where it has been used to provide an explanation for the Quantum Hall effect [4]). In mathematics, besides casting a new light on a familiar subject (geometry of manifolds), it may be used in many situations where the usual tools of differential geometry fail to apply because the space under study is not a «good» manifold (orbifolds, space of leaves of a foliation ...) or because there is no manifold at all (an abstract noncommutative algebra). Noncommutative geometry (and in particular cyclic cohomology) is also a good framework where to discuss infinite-dimensional spaces like the loop-space of a manifold M [30] and this brings us back again to the theory of strings.

Our aim, in what follows, is not to describe all the results and concepts of noncommutative geometry (a book would not be enough) but to describe a few topics which have been studied in the last five years. As already mentioned, the tools and techniques of noncommutative geometry are often the noncommutative counterparts of those of commutative (usual) geometry – although in many cases, one should not expect an obvious generalization! This remark motivates the organization of this paper. First, the algebra of functions on a manifold is replaced by an arbitrary associative – but not necessarily commutative – algebra A; then we introduce in Section 2 the universal differential algebra $\Omega(A)$ and in Section 3 the Hochshild cohomology H^* which plays the analogue of the algebra of differential forms on a manifold. The noncommutative counterpart of De Rham cohomology is cyclic cohomology (or better: periodic cyclic cohomology) and is described in Sections 4 and 5. This is replaced in some interesting cases by entire cyclic cohomology when the algebra is «big» (Section 7). Differential forms and the De Rham complex are not the only tools of differential geometry, one can indeed probe the structure of manifolds by studying fiber bundles above them, this leads to the definition of characteristic classes and to K-theory. The same thing is true here and we devote the last sections of this article to this study; however, these last topics will be only briefly discussed in order to keep the size of this paper reasonable.

The noncommutative analogue of vector bundles is described in Section 9 (and this leads into the definition of K-theory of algebras). Connections (Yang Mills fields) are a handy tool in usual geometry: the noncommutative corresponding concept is defined in the same section. Index theory for elliptic operators has also a noncommutative generalization which is described in Section 10. Finally, it can be seen that most ideas fit beautifully in the bivariant K-theory of Kasparov (KK-theory). This does not seem to be well-known by physicists and we conclude with a short introduction to the corresponding ideas.

Noncommutative geometry is a branch of mathematics which has undergone profound transformations in the last years but it does not seem to have achieved yet a development sufficient to allow for a nonperturbative description of a quantum field theory like quantum electrodynamics. Our feeling, however, is that it points in the right direction and our hope is to convince the theoretical physicist (for whom this review is written) that it is so.

How to read these notes

In the present paper, we follow an approach which does not necessarily coincides with the history of the subject (for instance, Fredholm modules are only introduced in section 10). The following wants to be a tentatively pedagogical introduction to noncommutative geometry (and also tries as often as possible to make the link with standard tools used by theoretical physicists). For this reason, several interesting points, although logically situated in the body of the paper, should certainly be skipped by the novice. The reader should first look at the following sections (in this order) and skip the others:

2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 3.1, 3.5, 3.6, 3.7, 3.9, 4.1, 4.5, 4.6, 5.1, 5.3, 5.4, 7.1 (ii), 9.1, 9.2, 9.4, 9.5, 10.1, 10.2, 10.3, 10.5, 12.

The special symbol ♠ has been added to the title of those sections that should be skipped on first reading.

Remarks and acknowledgements

It is believed that the formalism of non commutative geometry will someday be of fundamental importance in order to formulate our ideas about Quantum Field Theory. The physicist reader may be disapointed because this will not be done in the present review .... On the other hand, the mathematician reader has already several review articles at his disposal [8], [11], besides the basic reference [7]. Our aim, here, is mainly to help the reader interested in non commutative geometry and to narrow the gap that may exist between the standard mathematical concepts known (and used) by most theoretical physicists and those that appear often as a prerequisite for the reading of articles such as [7]. For this reason, many results will be given without proof (or only with an indication of what the proof is) but we will try to put the accent on what the motivations for the introduction of these new concepts are. Also, we will use, as often as possible, «standard» geometry as a guideline. As already mentionned previously, the following should be considered as an invitation to further study.

Most of what follows arose from discussions with A. Connes, J. Cuntz, D. Kastler, P. Seibt and R. Zekri. I would like to thank them here.

2. THE DIFFERENTIAL ENVELOPE(S) OF AN ASSOCIATIVE ALGEBRA

2.1. Definition

Let A be an associative algebra. We can associate with it a bigger algebra $\Omega(A)$, called its differential envelope, thanks to the following construction. To every element a in A, we associate a symbol δa. $\Omega(A)$, as a vector space is defined as the linear span of «words» built out of symbols such as a and δa. The multiplication in $\Omega(A)$ should satisfy the usual properties (associativity, distributively over +) (2) but we impose the following relation:

$$\delta a.b = \delta(ab) - a.\delta b. \tag{1}$$

The above relation allows us to shift symbols like b to the left and to write any element of $\Omega(A)$ as a linear combination of monomials of the kind $a_0 \delta a_1 \ldots \delta a_n$ or $\delta a_0 \delta a_1 \ldots \delta a_n$ where the a_i belongs to A. Let us work out one example

$$\begin{aligned} a_0 \delta a_1 \delta a_2 a_3 \delta a_4 &= a_0 \delta a_1 \delta(a_2 a_3) \delta a_4 - a_0 \delta a_1 a_2 \delta a_3 \delta a_4 \\ &= a_0 \delta a_1 \delta(a_2 a_3)\delta a_4 - a_0 \delta(a_1 a_2)\delta a_3 \delta a_4 + \\ &+ a_0 a_1 \delta a_2 \delta a_3 \delta a_4 . \end{aligned} \tag{2}$$

Notice that, in relation (1), $\delta(ab)$ is just a symbol (the symbol that we associate with the element ab of A in the construction of $\Omega(A)$). However, we want δ to become an operator, and this is done by defining

$$\begin{aligned} &\delta(a_0 \delta a_1 \delta a_2 \ldots \delta a_n) = \delta a_0 \delta a_1 \ldots \delta a_n \\ &\delta(\delta a_0 \delta a_1 \ldots \delta a_n) = 0 . \end{aligned} \tag{3}$$

Notice that (3) implies $\delta^2 = 0$.

$(\Omega(A), \delta)$ is then a differential algebra and δ is an odd derivation .This algebra is in particular Z-graded: $\Omega(A) = \oplus_{n=0}^{\infty} \Omega(A)^n$ where $\Omega(A)^0 = A$ and $\Omega(A)^p$ denotes the linear span of monomials $a_0 \delta a_1 \ldots \delta a_p$ or $\delta a_1 \ldots \delta a_p$. Since it has no more relations than those coming from A and from the Leibniz rule, the fact that it is a universal object is not too surprising. Here, by «universal» we mean that it factorizes derivations: if B is an algebra (it can be A itself), if α is a morphi sm from A to B (it can be the identity) and if D is a derivation from A to B twisted by α, *i.e.*,

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(2) More precisely, we could define $\Omega(A)$ as the free algebra generated by the symbols a, δa, $a \in A$, modulo the relation (1) and the relations $\lambda.a + \mu.b = (\lambda a + \mu b)$, $a.b = (ab)$, $\lambda.\delta a + \mu.\delta b = \delta(\lambda a + \mu b)$, where . and + denote the product and sum in the free algebra.

$D(ab) = D(a)\alpha(b) + \alpha(a)D(b)$ then, there exist a morphism $\bar{\alpha}$ from $\Omega(A)$ to B such that $\bar{\alpha}\ (a_0\delta a_1 \dots \delta a_n) = \alpha(a_0)D(\alpha(a_1))\dots D(\alpha(a_n))$. In particular, if B is a differential algebra and if there is a morphism from A to B, then B can be gotten from $\Omega(A)$ by a homomorphism of differential algebra. These properties can be summarized by the diagrams

$$\begin{tikzcd} & \Omega(A) \arrow[dr, "\hat{\alpha}"] & \\ A \arrow[ur] \arrow[rr, shift left, "\alpha"] \arrow[rr, shift right, "D"'] & & B \end{tikzcd}$$

and

$$\begin{tikzcd} & \Omega(A) \arrow[dr, "\hat{\alpha}"] & \\ A \arrow[ur] \arrow[rr, "\alpha"] & & (B,d) \end{tikzcd}$$

with $D = d\alpha$. These properies justify to call $\Omega(A)$ the «universal differential envelope» of A.

2.2. Problems of unit

The reader will have noticed that, for the moment, we did not mention the existence of a possible unit in A. In particular, if A is unital, we will call e its unit (*not* 1) and its differential in $\Omega(A)$ will not vanish ($\delta e \neq 0$). But then $\Omega(A)$ has no unit ($e\delta e \neq \delta e$) , and we can fix that by adding formally a unit (that we call 1) to $\Omega(A)$, with the rule $\delta 1 = 0$. The resulting algebra (the unital differential envelope of A) will be called $\tilde{\Omega}(A)$. Of course, we could also have built it by first adding formally a unit 1 to A (the resulting algebra being $\tilde{A}$), then constructing $\Omega(\tilde{A})$ with the extra rule $\delta 1 = 0$. Of course $\Omega(\tilde{A}) = \tilde{\Omega}(A)$. Notice that an arbitrary element of $\tilde{\Omega}(A)$ can be written canonically as a sum of monomial of the form $\tilde{a}_0 \delta a_1 \delta a_2 \dots \delta a_n$ where $\tilde{a}_0 = \lambda + a_0$, $\lambda = \lambda \cdot 1 \in \mathbb{C}$ and $a_i \in A$.

2.3. Other constructions of $\Omega(A)$

The construction of $\tilde{\Omega}(A)$ described above (let us call it construction No. 1) is enough for calculational purposes but there exist alternative constructions which one should be aware of, either because they are frequently used in the mathematical literature or because they cast another light on the nature of this universal object. We will present two other methods. Construction No. 2: Call $\tilde{\Omega}_0(A) = \tilde{A}$, $\tilde{\Omega}_n(A) = \tilde{A} \otimes A \otimes \dots \otimes A$, with n factors A, $n \neq 0$. Then define $\tilde{\Omega}(A) = \oplus_n \tilde{\Omega}_n(A)$. Notice that, since $\tilde{A} = \mathbb{C} + A$, we have $\tilde{\Omega}_n(A) = C \otimes A \otimes \dots \otimes A \oplus A \otimes A \otimes \dots \otimes A$. This shows that $\tilde{\Omega}_n(A)$ is isomorphic, as a vector space, with $A^{\otimes n} \oplus A^{\otimes n+1}$. In particular, $\tilde{\Omega}(A)$

cannot be identified with the tensor algebra over A. Of course, the link with construction No. 1 is made via the following correspondence: $\delta a_1 \delta a_2 \dots \delta a_n = 1 \otimes a_1 \otimes \dots \otimes a_n$ and $a_0 \delta a_1 \delta a_2 \dots \delta a_n = a_0 \otimes a_1 \otimes \dots \otimes a_n$, $a_i \in A$. Also, to be compatible with construction No. 1, multiplication from the right by an element of A is defined by the following rule:

$$(a_0 \otimes a_1 \otimes \dots \otimes a_n)b = a_0 \otimes a_1 \otimes \dots \otimes a_n b + \\ + \sum_{i=0}^{n} (-1)^{n-i} a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n \otimes b$$

multiplication in $\tilde{\Omega}(A)$ is specified by requiring that

$$(a_0 \otimes a_1 \otimes \dots \otimes a_n)(b_0 \otimes b_1 \otimes \dots \otimes b_m) = \\ = ((a_0 \otimes \dots \otimes a_n)b_0) \otimes b_1 \otimes \dots \otimes b_m.$$

Of course multiplication from the left by an element of A is just gotten by

$$b(a_0 \otimes a_1 \otimes \dots a_n) = ba_0 \otimes a_1 \otimes \dots \otimes a_n.$$

Construction No. 3: We first consider the multiplication m as a map from $\tilde{A} \otimes \tilde{A}$ to $\tilde{A}$ by $m(a \otimes b) = ab$. Then call $\tilde{\Omega}_0(A) = \tilde{A}$, $\tilde{\Omega}_1(A) = \text{Ker } m$ (notice that in the example $A = C(X)$, Ker m is the space of functions of two variables which vanish on the diagonal *i.e.*, such that $f(x,x) = 0$). Then define $\tilde{\Omega}_n(A) = \text{Ker } m \otimes_A \dots \otimes_A \text{Ker } m$ and $\tilde{\Omega}(A) = \oplus_n \tilde{\Omega}_n(A)$. Notice that the previous tensor products are taken over A and not over $\mathbb{C}$ as it was before. This last definition of the differential envelope seems more involved than the previous one but it is rather convenient as we shall see later. Here again, monomials like $\delta a_1 \delta a_2 \dots \delta a_n$ and $a_0 \delta a_1 \dots \delta a_n$ can be written in terms of tensor products but the correspondence is not the same as in the construction No. 2, also the product rule is different. For instance, we may write $\delta b = 1 \otimes b - b \otimes 1$, $a\delta b = a \otimes b - ab \otimes 1$; it is clear that δb and $a\delta b$ belong to Ker m since $m(a \otimes b - ab \otimes 1) = ab - ab = 0$. In terms of tensor products, the product rule is now gotten by concatenation, for instance:

$$a\delta b \delta c = (a \otimes b - ab \otimes 1)(1 \otimes c - c \otimes 1) \\ = a \otimes b \otimes c - ab \otimes 1 \otimes c - a \otimes bc \otimes 1 + ab \otimes c \otimes 1.$$

We can go from this expression to the corresponding one in construction No. 2 by killing the «1» which are not in first position.

2.4. The acyclic Hochshildoperator

The main thing to remember is that, whatever the way we choose to realize $\tilde{\Omega}(A)$, the calculation rules are those given at the beginning of this section. Also, it is clear that $\tilde{\Omega}(A)$ comes equipped with a Z-grading which counts the number of δ. Since any element of $\tilde{\Omega}_n(A)$ can be written as a linear combination of monomials of the kind $\tilde{a}_0\delta a_1 \ldots \delta a_n$ and $\delta a_1 \ldots \delta a_n$, it is convenient to write $\tilde{\Omega}(A) = \tilde{A}\delta\Omega(A) + \delta\Omega(A)$. Since we have an operation δ of square zero, it is natural to compute the cohomology of the complex $\tilde{\Omega}_*(A)$. From the construction of δ, it is *a priori* clear that this cohomology is just zero. This can also be seen as follows: let ω be a monomial of grade $\partial\omega$ and x an element in A; we define the following operator in $\Omega(A)$ (not in $\tilde{\Omega}(A)$ since x is not determined by δx if $\delta 1$ is zero).

$$\beta'(\omega\delta x) = (-1)^{\partial\omega}\omega x. \tag{4}$$

Notice that β' is defined in $\Omega(A)$ but not in $\tilde{\Omega}(A)$ since $\delta(x)$ and $\delta(x+1)$ are the same in $\tilde{\Omega}(A)$. Then, rules of calculation in $\tilde{\Omega}(A)$ show that $\beta'\delta + \delta\beta' = 1$; therefore, for any τ we get $\beta'\delta\tau + \delta\beta'\tau = \tau$ and if $\delta\tau = 0$ we get $\tau = \delta\beta'\tau$ which shows that the cohomology of δ is indeed trivial (in $\tilde{\Omega}(A)$, this cohomology is almost trivial since $\delta 1 = 0$ although 1 is not the δ of something). The operator β' is called the acyclic Hochshild homology operator for reasons which will become clear in the next section.

2.5. The Z_2-Graded Case

It may happen that A is a Z_2-graded algebra. In this case, we may forget the Z_2-graduation and construct the differential envelope as above, however, we may also use this graduation and build a universal Z_2-graded differential algebra associated with A that we propose to call the differential superenvelope. The only difference is that now, eq. (1) is replaced by the following

$$\delta a.b = \delta(ab) - (-1)^{\partial a} a.\delta b \tag{5}$$

where ∂a denotes the intrinsic Z_2-grading of $a \in A$. The differential superenvelope is also a universal object because it factories graded derivations. In what follows, we will mostly give formulae valid in the ungraded case in order not to clutter them with minus signs. Some of the formulae are anyway the same, for instance eq. (4), but now $\partial\omega$ denotes the total grading of ω, *i.e.*, the sum of the $Z-$ grading and Z_2-grading.

The construction of the differential envelope goes back to Cartan but it can be found in [6,7] under several disguises, let us also quote [8,9] for a particular study of the Z_2-graded case.

2.6. Example 1: $A = C(X)$

In order to illustrate the previous constructions, let us contemplate the classical example of a commutative algebra $A = C(X)$ where $C(X)$ will denote in the following of this paper the space of smooth functions on a differentiable manifold X (it may happen that some properties that we discuss here and after have to be modified if the space is bigger than $C^\infty(X)$, so we will stick to the smooth case for simplicity but will most of the time drop the ∞ superscript). The first thing to notice is that $A \otimes A = C(X \times X)$ and (3) more generally $A^{\otimes n} = C(X \times \ldots \times X)$, therefore, in a sense, when we go from A to $\Omega(A)$ we go from a one-body problem to a many-body problem. We already identified $\Omega_1(A)$ with the functions of two variables which vanish on the diagonal (of Description No. 3). Indeed if $f \in A$, then, since $\delta f = 1 \otimes f - f \otimes 1$, we get $\delta f(x,y) = f(y) - f(x)$ and we can therefore visualize δf as a finite difference. The Leibniz rule $\delta(fg) = \delta f g + f \delta g$ can be translated as follows

$$
\begin{aligned}
&f(y)g(y) - f(x)g(x) = \\
&= [f(y) - f(x)]g(y) + f(x)[g(y) - g(x)].
\end{aligned}
$$

More generally, elements of $\Omega_1(A)$ will be of the kind $F(x,y) = G(x,y) - G(x,x)$ where G is an arbitrary function on $X \times X$. In the same way, if $f, g, h \in A$, we get

$$f\delta g \delta h(x,y,z) = f(x)(g(y) - g(x))(h(z) - h(y))$$

and more general elements of $\Omega_2(A)$ will be of the kind

$$F(x,y,z) = G(x,y,z) - G(x,x,z) - G(x,y,y) + G(x,x,y)$$

where G is an arbitrary function on $X \times X \times X$. Notice that such functions vanish on the diagonals (1–2) and (2–3) but not (1–3). This can obviously be generalized. $\Omega_n(A)$ consists of those functions on $X \times \ldots \times X$ ($n+1$ factors), which vanish on contiguous diagonals. To illustrate the universal property of $\Omega(A)$, we may consider the differential algebra $\Lambda(X)$ of differential forms over X; we have a map from A to $\Lambda^0(X)$ which is just the identity i. Then, there is a universal covering homomorphism $\bar{\imath}$ from $\Omega(A)$ to $\Lambda(X)$ such that $\bar{\imath}\,(f_0 \delta f_1 \ldots \delta f_n) = f_0 df_1 \wedge df_2 \wedge \ldots \wedge df_n$. Of course, the kernel of this map is rather large: this is already clear from the fact $\Lambda(X)$ has dimension $2^{\dim X}$ but $\Omega(A)$ is infinite dimensional; also, if ω is a «1-form» in

(3) Actually, we should use an inclusion sign rather than an equal sign but the equality can be made true after completion.

$\Omega(A)$ – *i.e.*, an element of $\Omega_1(A)$, its image under $\bar{\imath}$ .is a one form $\theta(\omega)$ in $\Lambda(X)$ and is such that $\bar{\imath}(\omega) \wedge \bar{\imath}(\omega) = 0$, however ω^2 is not zero but a well-defined element of $\Omega_2(A)$. This suggests that noncommutative connections will have new unexpected features, even in the «classical» case where $A = C(X)$, cf. Section 9.4.

When X is a Riemannian manifold, we can also consider the example where B is the Clifford algebra of the tangent bundle of X. B is not a differential algebra but $\not{d} : f \in C(X) \to \not{d}f = \gamma^\mu \partial_\mu f$ is a derivation from $A = C(X)$ to B. Again A can be identified with a vector subspace of B and $\bar{\imath}(f_0 \delta f_1 \ldots \delta f_n) = f_0 \not{d}f_1 \cdots \not{d}f_n$.

2.7. Example 2: $A = \mathbb{C}$

In the particular case where X is just a point, the complex algebra of $C(X)$ is just the algebra of complex numbers. Let us describe $\tilde{\Omega}(\mathbb{C})$, the differential envelope of $\mathbb{C}$ (cf. also [8]). According to Section 2.2, let us call e the unit of $\mathbb{C}$ and let us add an extra unit 1. Elements of $\tilde{\Omega}(\mathbb{C})$ are linear combinations of monomials $\lambda\delta e + \delta e$ or $\mu e\delta e \ldots \delta e$. We have the rule $\delta(e^2) = e\delta e + \delta e e$ but $e^2 = e$ therefore $\delta e = e\delta e + \delta e e$. We have now an interesting representation of $\tilde{\Omega}(\mathbb{C})$ in terms of creation and annihilation operators. Indeed, let $\mathcal{H}$ be a Hilbert space (actually w e will take $\mathcal{H} = \mathbb{C}$) and $\mathcal{F}$ the bosonic or fermionic Fock space associated with $\mathcal{H}$; let also $f = 1 \in \mathcal{H}$ and just call $a \doteq a(1)$, $a^+ \doteq a^+(1)$ the annihilation and creation operators associated with $f = 1$. Then it is clear that we may represent e as the projector $\frac{1}{2}(1+\gamma)$ where $\gamma \doteq (-1)^{a^+a}$ counts the parity of the number of particles, and δe as the annihilation operator a; indeed it is easy to check that

$$a = \left(\frac{1+\gamma}{2}\right) a + a \left(\frac{1+\gamma}{2}\right).$$

Notice that a monomial of $\tilde{\Omega}(\mathbb{C})$ will be represented as

$$\left(\lambda + \mu \left(\frac{1+\gamma}{2}\right)\right) a^p$$

where $\lambda,\ \mu \in \mathbb{C}$ and $p \in \mathbb{N}$. Setting $q \doteq \delta e$, the defining relations of $\tilde{\Omega}\mathbb{C}$ read $\gamma q + q\gamma = 0$, $\gamma^2 = 1$.

2.8. The Cuntz and Zekri algebras ♠

Before ending this section we want to mention that there exist another «universal» object associated with any arbitrary associative algebra A, namely the Cuntz algebra $Q(A)$ (which is sometimes denoted qA). This object was defined abstractly in [10] but it was realized afterwards (cf. [20, 23]) that $Q(A)$, as a set, is nothing else than the differential envelope $\Omega(A)$, however the product law is not the same. Let us indeed choose α, β in $\Omega(A)$ and define

$$\alpha o \beta = \alpha\beta \qquad \text{if } \alpha \text{ is even } (i.e., \quad \alpha \in \Omega_{2n}(A))$$

$$\alpha o \beta = \alpha\beta + \lambda\alpha\delta\beta \qquad \text{if } \alpha \text{ is odd } (i.e., \quad \alpha \in \Omega_{2n+1}(A).$$

In (6), λ is an arbitrary scalar parameter. We call $Q(A)$ the set $\Omega(A)$ endowed with the new product; notice that if $\lambda = 0$, then both algebras are the same. The highly nontrivial remark (although it takes three lines to prove it) is that $\circ$ is an associative product: $\alpha \circ (\beta \circ \gamma) = (\alpha \circ \beta) \circ \gamma$. Therefore $Q(A)$ is an associative algebra and appears as a deformation of $\Omega(A)$; in what follows let us choose $\lambda = -1$. Let a and b be two elements of A, then from (1) and (6) we get

$$\delta(a \circ b) = \delta a \circ b + a \circ \delta b + \delta a \circ \delta b. \tag{7}$$

In the cases where there is no risk of confusion, we may call «.» the product in $Q(A)$ and «q» the differential δ (to remind us where we are!). Then (7) reads

$$q(a.b) = qa.b + a.qb + qa.qb. \tag{8}$$

Notice that q appears naturally as an infinitesimal homomorphism. Indeed if u is a map satisfying $u(ab) = u(a)u(b)$, then setting $u = 1 + q$ leads directly to eq. (8). Like $\Omega(A)$, the algebras $Q(A)$ is the linear span of monomials of the type $qa_1 \dots qa_n$ or the type $a_0 qa_1 \dots qa_n$ and we may write $Q(A) = Aq(A) \oplus q(A)$. Calling $\Omega_p = \Omega(A)_p$ and $Q_p = Q(A)_p$, we notice that $Q(A)$ is no longer graded but only filtered: we have $\Omega_p\Omega_q \not\subseteq \Omega_{p+q}$ but $Q_pQ_q \subset \oplus_{i \leq p+q} Q_i$, so that $Q(A)$ bears the same analogy with $\Omega(A)$ as the Clifford algebra compared with the algebra of exterior forms. The main interest of the Cuntz algebra $Q(A)$ is that it factorizes pairs of homomorphisms. Let indeed φ and ψ be two homomorphisms from the algebra A into the algebra B then $\kappa \doteq \psi - \varphi$ is not a homomorphism, indeed $\kappa(ab) = \varphi(a)\kappa(b) + \kappa(a)\varphi(b) + \kappa(a)\kappa(b)$ (in particular if $A = B$ and $\varphi = 1$ then κ satisfies the same properties as q). Obviously we may replace the data (φ, ψ) by the data (φ, κ). Universality of $Q(A)$ means that there exist a morphism ν from $Q(A)$ to B such that $\varphi(a) = \nu(a)$ and $\kappa(a) = \nu(qa)$.

This can be summarized by the following diagram

$$\begin{tikzcd} & Q(A) \arrow[dr, "\nu"] & \\ A \arrow[ur] \arrow[rr, shift left, "\varphi"] \arrow[rr, shift right, "\psi"'] & & B \end{tikzcd}$$

This property allows a very simple definition of the Kasparov KK^0-group or for that matters K^0-groups, cf. Section 11. Besides, as we shall see later when we describe cyclic cohomology, it is sometimes easier to use QA than ΩA. A last universal object that one can associate [28] with an arbitrary associative algebra A is the

Zekri algebra $\epsilon(A) = Q(A) \times_\sigma Z_2$. Here σ denotes the following involutive automorphism: $\sigma(x^0 qx^1 \dots qx^n) = (-1)^n(x^0 - qx^\circ)qx^1 \dots qx^n$; $\sigma(qx^1 \dots qx^n) = (-1)^n qx^1 \dots qx^n$. $\epsilon(A)$ is then defined as the cross-product of $Q(A)$ and Z_2 , *i.e.*, one identifies $(\omega, \pm 1)$ with $(\sigma(\omega), -(\pm 1))$. As a vector space $\epsilon(A) = Q(A) \oplus Q(A)$ but as an algebra, it can be described as a Z_2-graded subalgebra of the space of 2×2 vertices with elements in $Q(A)$:

$$\omega_0, \omega_1 \in Q(A) \to \begin{bmatrix} \omega_0 & \omega_1 \\ \sigma(\omega_1) & \sigma(\omega_0) \end{bmatrix} \in \epsilon(A);$$

the grading is of course given by

$$F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This algebra can be used to describe cyclic cohomology: as we shall see later, cyclic cocycles will be related to «graded traces» on $\Omega(A)$ but to usual traces on $Q(A)$ or $\epsilon(A)$ depending upon the parity of the cocycle. $\epsilon(A)$ was first introduced in [24] to allow a simple definitions of the KK ^{1} groups (cf. Section 11).

2.9. Exterior differential forms and derivations ♠

In the case where $A = C^\infty(X)$, we know that differential forms can also be defined as A-valued antisymmetric R-multilinear forms acting on vector fields; the vector fields themselves can be defined as derivations of the commutative algebra A . Let us mimic the above construction in the noncommutative set-up. A being an associative algebra, let $L = \mathrm{Der} A$ be the space of derivations on A ($\xi \in L \Leftrightarrow \forall a$, $b \in A$, $\xi(ab) = \xi(a)b + a\xi(b)$) ; L is the noncommutative analog of the space of vector fields. Let $\Lambda^*(L, A)$ be the space of antisymmetric multilinear forms on L and valued in A *i.e.*,

$$\begin{aligned} &\lambda(\xi_1, \dots, \xi_{i-1}, \xi_{i+1}\xi_i, \xi_{i+2}, \dots, \xi_n) = \\ &= -\lambda(\xi_1, \dots, \xi_{i-1}, \xi_i, \xi_{i+1}, \xi_{i+2}, \dots \xi_n). \end{aligned}$$

The wedge product $\wedge$ and the differential d on $\Lambda^*(L, A)$ are then defined exactly as in the standard commutative case; for example

$$\lambda \wedge \mu = \frac{(n+m)!}{n!m!} A(\lambda \otimes \mu)$$

where A is the antisymmetriser; also if $a \in A$, we define $(da)(\xi) = \xi(a)$ and extend it to the whole of Λ^*.

$$d\lambda(\xi_1,\xi_2,\ldots,\xi_{n+1})=\sum_{i=1}^{n+1}(-1)^{i+1}\xi_i\omega(\xi_1,\ldots,\hat{\xi}_i,\ldots,\xi_{n+1})$$
$$+\sum_{\substack{i,j=1\\ i\leq j}}^{n}(-1)^{i+j}\lambda([\xi_i\xi_j],\xi_1,\ldots,\hat{\xi}_i,\ldots\hat{\xi}_j,\ldots,\xi_{n+1})).$$

We could call $\Lambda^*(L,A)$ the algebra of exterior differential forms on A. Notice that by universality of the algebra $\Omega(A)$ (cf. Section 2.1), we have a morphism $a_0\delta a_1\ldots\delta a_n\in\Omega(A)\to a_0 da_1\wedge\ldots\wedge da_n\in\Lambda(L,A)$ mapping the «universal differential forms» onto the «exterior differential forms». Notice, that the complex $\Lambda^*(L,A)$ can be considered as a particular case of the complex $\Lambda^*(L,E)$ of Chevalley-Eilenberg, where L is a Lie algebra and E is a module over L. On the other hand, $L = Der(A)$ is not usually a A-module (i.e., if $\xi\in L$ and $a\in A$, then $a\xi$ is not usually a derivation). Moreover, even if it is a A-module and if $\omega\in\Lambda^*(L,A)$ and $a\in A$, then $\omega(a\xi)$ is not necessarily equal to $a\omega(\xi)$; it is therefore natural to introduce $\Lambda_A(L,A)\subset\Lambda(L,A)$ by requiring A-linearity rather than only $\mathbb{C}$-linearity (in the case where L is a A-module); in this last case, one can even be more restrictive and define $\Lambda_{DR}(L,A)$ as the subalgebra of $\Lambda_A(L,A)$ linearly generated by totally decomposable tensors: in this way we obtain the classical De Rham complex in the case $A = C^\infty(X)$. The above construction can be nicely generalized to the case where A is Z_2-graded and is particularly nice when A is graded commutative algebra (we could consider A as an algebra of functions over a superspace), cf. [8] [17].

Although the algebra $\Lambda(L,A)$ plays an important role in the study of commutative differential geometry, its importance, in the noncommutative content is weakened by the fact that many algebras have no derivations at all (for example the complex numbers), in those cases, the universal map $\Omega(A)\to\Lambda(L,A)$ is just zero! This is not therefore the right way of introducing differential forms in the most general non commutative framework. Nevertheless, we will see in Section 3 that it is possible to define for any algebra A a complex $H^*(A)$ called the Hochshild complex which plays in all cases the role of the complex of differential forms. At a later stage (Section 4) we will introduce an operator B on $H^*(A)$ which will play the role of the De Rham boundary. In the cases where the algebra A has enough derivations, it becomes usefull to consider $\Lambda(L,A)$, cf.[47].

3. HOCHSHILD (CO)-HOMOLOGY OF AN ASSOCIATIVE ALGEBRA

3.1. Motivations

In the simple case of «commutative geometry», we start from a commutative algebra A (usually the algebra of functions over some manifold X). We have already constructed the differential envelope $\Omega(A)$ but we should be able to construct also the algebra $\Lambda(X)$ of differential forms, in a way which can be generalized to the non-commutative case (and therefore in a way which does not make explicit reference to the manifold X). Roughly speaking, as we shall see below the Abelian groups $\Lambda^*(X)$ will *appear* as the Hochshild homology groups $H_*(A)$ of the algebra A . There are several equivalent definitions of Hochshild (co)-homology and it is useful to look at several of them in order to read and use the existing literature.

3.2. Hochshild cohomology of an algebra A with coefficients in a bimodule $\mathcal{M}$ ♠

This definition is slightly too general for our purposes but it is standard. We consider the space $C^n(A,\mathcal{M})$ of n-linear maps $T(a_1,a_2,\ldots,a_n)$ from an algebra A into a bimodule $\mathcal{M}$ (*i.e.*, we know how to multiply from the left and from the right by elements of A). Then we call (bT) the $(n+1)$-linear map:

$$\begin{aligned}[bT](a_1,\ldots,a_{n+1}) =&a_1T(a_2,\ldots,a_{n+1})+\\ &+\sum_{i=1}^{n}(-1)^iT(a_1,\ldots,a_ia_{i+1},\ldots,a_{n+1})+\\ &+(-1)^{n+1}T(a_1,\ldots,a_n)a_{n+1}.\end{aligned} \tag{9}$$

It is easy to see that the operator b has square zero and we define the Hochshild cohomology groups $H^*(A,\mathcal{M})$ as the cohomology of the complex $C^*(A,\mathcal{M})$.

3.3. Hochshild cohomology of an algebra A ♠

This is the definition that we are interested in and it can be given either as a particular case of the previous one, or directly. As a particular case of 3.2, we may consider $\mathcal{M}$ as the (algebraic) dual A^* of A . It is indeed a bimodule since if $a,b\in A$ and $\varphi\in A^*$ we have $a\varphi b\in A^*$ defined by $(a\varphi b)(c)=\varphi(bca)$. Now, we may consider elements $T\in C^n(A,A^*)$ as $(n+1)$ linear forms on A :

$$\tau(a_0,a_1,\ldots,a_n)\doteq[T(a_1,\ldots,a_n)](a_0)\ \in \mathbb{C}. \tag{10}$$

Rather than defining the coboundary operator b in terms of T , we can therefore do it directly in terms of τ :

$$\begin{aligned}[b\tau](a_0,\ldots,a_{n+1}) =&\tau(a_0a_1,a_2,\ldots,a_{n+1})+\\ &+\sum_{i=1}^{n}(-1)^i\tau(a_0,\ldots,a_ia_{i+1},\ldots,a_{n+1})+\\ &+(-1)^{n+1}\tau(a_{n+1}a_0,\ldots,a_n).\end{aligned} \tag{11}$$

As a particular case of 3.2, we call $C^{n+1}(A)\doteq C^n(A,A^*)$ and $H^{*+1}(A)\doteq H^*(A,A^*)$ where we no longer mention the bimodule A^* .

3.4. Hochshild homology of A in the tensorial algebra ♠

We may consider n-linear forms on A as 1-linear forms on $A^{\otimes n}$, therefore, by duality, we define the Hochshild homology operator $\beta : A^{\otimes n+1} \to A^{\otimes n}$ as

$$\beta(a_0 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, \ldots \otimes a_i a_{i+1} \otimes \ldots a_n) + \\ + (-1)^n (a_n a_0 \otimes \ldots \otimes a_{n-1}). \tag{12}$$

Of course, the operator β has square zero and the homology groups are denoted by $H_*(A)$.

3.5. Hochshild homology and cohomology of A defined in the differential envelope $\Omega(A)$

Let us add a unit 1 to A and call $\tilde{A} = A + \mathbf{C}1$ the unital extension of A. The Hochshild homology of $\tilde{A}$ can be defined as in 3.4, however, it can be shown [11] that one does not loose anything by considering the subcomplex of $\tilde{\Omega}_n \doteq \tilde{A} \otimes A^n$ generated by elements of the kind $a_0 \otimes a_1 \ldots \otimes a_n$ where 1 can only be in first position. This suggests that Hochshild homology of A can be defined directly within the differential envelope $\tilde{\Omega}(A)$ (as it is done in [8]). Moreover, if we represent the element $a_0 \delta a_1 \ldots \delta a_n$ of $\tilde{\Omega}(A)$ by $a_0 \otimes a_1 \ldots \otimes a_n$, we see, using (1), than (4) is almost equal to (12) except for the last flip-over term. The Hochshild boundary operator β can then be defined directly as follows (compare with 2.4) in $\tilde{\Omega}_*(A)$. Let ω be a monomial of grade $\partial\omega$ and x an element in A, we set

$$\beta(\omega\delta x) = (-1)^{\partial\omega}[\omega, x] \\ \beta(x) = 0. \tag{13}$$

It may be useful to introduce the «flip-over» operator α

$$\alpha(\omega\delta x) = (-1)^{\partial\omega} x\omega. \tag{14a}$$

Then $\beta = \beta' - \alpha$ where β' was introduced in Section 2.4. Notice that $\beta^2 = \beta'^2 = 0$ but the homology of β' is trivial (as the one of δ, and for the same reason, cf. Section 2.4). We could think that the Hochshild coboundary operator b is defined by taking the transpose β^t acting on forms over the differential envelope; although this is possible (and done by several authors), the cohomology that we would get is slightly «too big» (in a sense we would be over counting the complex numbers: this comes from the fact that it is enough to know the value of a form φ on elements $a_0 \delta a_1 \ldots \delta a_n \sim a_0 \otimes a_1 \ldots \otimes a_n$ and not necessarily on $\delta a_1 \ldots \delta a_n \sim 1 \otimes a_1 \ldots \otimes a_n$, with $1 \in \mathbf{C}$).

In such a case, it could be convenient to define in turn «reduced-Hochshild cohomology groups.» Rather than doing that, we follow [7] or [8] and define the Hochshild complex as follows: first the Hochshild co-chains of degree n will be those forms φ or $\Omega^n(A)$ which vanish on the part $\delta\Omega^n(A)$ (remember that $\Omega(A) = A\delta\Omega(A) + \delta\Omega(A)$) *i.e.* $\varphi \circ \delta = 0$, then the coboundary operator b is defined as the transposed of β restricted to $A\delta\Omega(A)$, *i.e.*,

$$[\varphi \text{ is a Hochshild cochain}] \Leftrightarrow \forall\omega \in \Omega(A)\varphi(\delta\omega) = 0$$

(14b) $$[\varphi \text{ is a Hochshild cocycle}] \Leftrightarrow [\varphi \text{ is a Hochshild cochain and } \forall\omega \in \Omega(A), [b\varphi](\omega) \doteq \varphi(\beta(\epsilon\omega)) = 0]$$

where ϵ is the projector of $\Omega(A)$ onto $A\Omega(A)$. We warn the reader that the notation $H^*(A)$ for the Hochshild cohomology groups is unfortunately not standard (some authors call it $HH^*(A)$ or $H^*(A, A^*)$) and $H^*(A)$ sometimes denotes the cohomology of β^t (in which case $\bar{H}^*(A)$ denotes the reduced Hochshild cohomology). Often one says that φ is «closed» if it vanishes on $\delta\Omega(A)$; it should be remembered that it means «closed for δ » and usually not for b . In what follows, we should remember that there is a one to one correspondence between Hochshild cochains defined as $(n+1)$ forms φ on A or as forms $\hat{\varphi}$ on the subspace $\tilde{\Omega}^n A$ of the universal differential algebra $\tilde{\Omega} A$. More precisely

$$\hat{\varphi}(a_0\delta a_1 \ldots \delta a_n) = \varphi(a_0, a_1, \ldots, a_n)$$

where $a_i \neq 1$, $i > 0$. We will usually not distinguish between φ and $\hat{\varphi}$.

When the algebra A is Z_2 graded, one can define as follows the Hochshild operator

$$(b\varphi)(a_0, a_1, \ldots, a_n, a_{n+1}) = \sum_{j=0}^{n}(-1)^j\varphi(a_0, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) -$$
$$-(-1)^{\partial a_{n+1}\sum_{i=0}^{n}\partial a_i}\varphi(a_{n+1}a_0, a_1, \ldots, a_n)$$

where ∂a_i denotes the intrinsic Z_2 grade (0 or 1) of the element a_i .

The Hochshild dimension of an algebra A is defined as the integer p (possibly infinite) such that $H^n(A) = 0$ if $n > p$. As we shall see later, in the case where A is the algebra of functions on a manifold X , we have $p = \dim X$.

3.6. Hochshild cohomology of the algebra of complex numbers

This is a continuation of the example started in Section 2.7. Using eq. (13) and calling $q = \delta e$ (as in [8]), we find

$$\beta(e) = \beta(q^{2n+1}) = \beta(eq^{2n+1}) = 0, \qquad n \geq 0$$
$$\beta(eq^{2n}) = eq^{2n-1}, \qquad n \geq 1$$
$$\beta(q^{2n}) = (2e-1)q^{2n-1}, \qquad n \geq 1.$$

Therefore, at the homological level, we find immediately that $Z_0 = \mathbf{C}$, $Z_{2n} = 0$, $Z_{2n+1} = B_{2n+1} = \mathbf{C} + \mathbf{C}$ and therefore $H_0 = \mathbf{C}$, $H_n = 0$, $n \geq 1$. Notice that, at the cohomological level, one finds $Z^0 = \mathbf{C}$, $Z^{2n} = B^{2n} = \mathbf{C} + \mathbf{C}$ (or $\mathbf{C}$ if we use b rather than β^t) and $Z^{2n+1} = 0$. In any case, we get $H^0 = \mathbf{C}$, $H^n = 0$, $n \geq 1$.

As announced in 3.1, Hochshild homology should be thought of as the right generalization of the concept of differential forms. The above result for the algebra $A = \mathbf{C}$ should therefore not be too surprising in view of the fact that A is indeed the algebra of functions over a manifold which is just a single point (its algebra of differential forms is then essentially trivial, but in dimension zero).

3.7. Hochshild cohomology of the commutative algebra $A = C^\infty(X)$ where X is a smooth manifold

When $A = C^\infty(X)$, we already constructed the differential algebra $(\Omega(A), \delta)$. However this algebra cannot be identified with the algebra exterior forms $\Lambda^*(X)$ – although there exists a universal morphism from the former to the latter. Indeed $\Omega(A)$ is «too big», it is infinite-dimensional, whereas, taking for example $X = R^n$, we get $dim(\Lambda^* X) = 2^n$ In other words, the kernel of the universal map is rather large. For example, the following element $\beta(\omega)$ is nonzero in $\Omega(A)$ but its image in $\Lambda(X)$ vanishes.

$$\omega = a_0 \delta a_1 \delta a_2 \delta a_3,$$

$$\beta(\omega) = [a_0 \delta a_1 \delta a_2, a_3]$$

$$\beta(\omega) = a_0 \delta a_1 \delta(a_2 a_3) - a_0 \delta(a_1 a_2) \delta a_3 + \\ + a_0 a_1 \delta a_2 \delta a_3 - a_0 a_3 \delta a_1 \delta a_2 \neq 0$$

but

$$a_0 da_1 \wedge d(a_2 a_3) - a_0 d(a_1 a_2) \wedge da_3 + \\ + a_0 a_1 da_2 \wedge da_3 - a_0 a_3 da_1 \wedge da_2 = 0.$$

As announced in 3.1, the complex of Hochshild homology groups $H_*(A)$ should play the role of the De Rham complex $\Lambda^*(X)$ of differential forms on X. At the cohomological level, we have of course a dual situation and $H^*(A)$ will appear as the complex of De Rham currents $\Lambda_*(X)$. De Rham currents are, in a sense, distributional forms [12,13]. They are the dual of forms. If C is a current and ω is a form, then $\langle C, \omega \rangle$ is a number. Notice that we should not talk (yet) of the «complex» of Hochshild groups since we have not (yet) defined any operator playing the role in the noncommutative case of the De Rham exterior derivative d. This will be done in the next chapter. To convince the reader that Hochshild cocycles indeed behave as De Rham currents, let us first consider the action of β on the monomial $f\delta g\delta h$ of $\Omega^2(A)$. As we saw in 2.6, this monomial can be thought of as a function on the manifold $X \times X \times X$ vanishing pairwise on contiguous diagonals:

$$[f\delta g\delta h](x,y,z) = f(x)(g(y) - g(x))(h(z) - h(y))$$

$$\begin{aligned}\beta(f\delta g\delta h) = (-1)^1[f\delta g, h] &= -f.\delta g.h + h.f.\delta g \\ &= -f\delta(gh) + fg\delta h + hf\delta g\end{aligned}$$

We get a function on $X \times X$ which reads explicitly

$$\begin{aligned}[\beta(f\delta g\delta h)](x,y) = &- f(x)(g(y)h(y) - g(x)h(x)) + \\ &+ f(x)g(x)(h(y) - h(x)) + \\ &+ h(x)f(x)(g(y) - g(x)) = \\ &= - f(x)(g(y) - g(x))(h(y) - h(x)).\end{aligned}$$

A further action of the operator β would give zero as it should (the last term becomes $h(x) - h(x)$). Now let C be a two-dimensional De Rham current, we may associate with it the following forms φ on Ω^2 – a distribution on X^3 :

$$\varphi(f\delta g\delta h) \doteq \langle C, fdg \wedge dh\rangle;$$

from the known properties of d and $\wedge$ let us show that $b\varphi = 0$

$$\begin{aligned}[b\varphi](f^0\delta f^1\delta f^2\delta f^3) &= \varphi(\beta(f^0\delta f^1\delta f^2\delta f^3)) = \\ &= \varphi([f^0\delta f^1\delta f^2, f^3]) = \\ &= \varphi(f^0\delta f^1\delta f^2 f^3) - f^0\delta(f^1 f^2)\delta f^3 + \\ &+ f^0 f^1\delta f^2\delta f^3 - f^0 f^3\delta f^1\delta f^2) = \\ &= \langle C, f^0 df^1 \wedge d(f^2 f^3)\rangle - \\ &- \langle C, f^0 d(f^1 f^2) \wedge df^3\rangle + \\ &+ \langle C, f^0 f^1 df^2 \wedge df^3\rangle - \langle C, f^0 f^3 df^1 \wedge df\rangle = \\ &= 0.\end{aligned}$$

More generally to any h-dimensional De Rham current C, one associates the following Hochshild cocycle $\varphi \in Z^k(A)$ (a distribution over X^{k+1})

$$\varphi(f^0 \delta f^1 \dots \delta f^k) \doteq \langle C, f^0 df^1 \wedge \dots \wedge df^k \rangle. \tag{15}$$

Conversely, given an element in $H^k(A)$, we choose a representative φ and define the De Rham current C as

$$\langle C, f^0 df^1 \wedge \dots \wedge df^k \rangle = \sum_{\sigma \in \varphi_k} \epsilon_\sigma \varphi(f^0 \delta f^{\sigma(1)} \dots \delta f^{\sigma(k)}). \tag{16}$$

(For a more refined and detailed treatment, cf. [7]). Considering φ as a distribution on X^{k+1}, we could also denote $\varphi(f^0 \delta f^1 \dots \delta f^k)$ by an expression like

$$\int_{X^{k+1}} \varphi(x_0, \dots, x_{k+1}) f^0(x_0) f^1(x_1) \dots f^k(x_k) \prod_i dx^i$$

and eq. (16) shows that the support of φ is contained in the diagonal of X^{k+1}.

3.8. Hochshild (co)homology of the algebra $\Lambda(X)$ of differential forms ♠

In the last subsection, we saw the Hochshild homology of the commutative algebra $C^\infty(X)$ was related to the set of differential forms $\Lambda(X)$; but this set, endowed with the laws of addition and exterior multiplication is itself a noncommutative (and Z_2-graded) algebra. It is therefore natural to study its own Hochshild (co)-homology. Let us remember (Section 2.5) that when the algebra A under study is Z_2-graded, one may construct two kinds of differential envelopes: the Z_2-graded one («superenvelope») and the nongraded one. In the Z_2-graded case, it is natural to define the Hochshild operator β, not by eq. (13) but by the following:

$$\beta(\omega \delta x) = (-1)^{\partial \omega} [\omega, x]_g \tag{17}$$

with

$$[\omega, x]_g = \omega x - (-1)^{\partial x \partial \omega} x \omega$$

where the commutator has been replaced by the graded commutator [8]. The corresponding (co)-homology is the called Z_2-graded Hochshild (co)-homology but to be consistent with the physicists's tradition we should call it «superhomology». Actually, one also introduces in the case of the algebra of differential forms, a Hochshild «hyperhomology» which uses the fact that $\Lambda(X)$ is not only an associative algebra but a differential algebra, so that $\Omega(\Lambda(X))$ can be equiped with two kinds of differentials (d and δ). The Hochshild hyperhomology of the algebra of differential forms is related to the usual homology of the free loop space of X which plays an important role in string theory. We refer to [14–16] for more details.

3.9. Remarks about differential forms in a noncommutative context ♠

1) In the commutative case, calling $A = C^\infty(X)$, X being some manifold of dimension n, we know that the space $\Lambda(X)(x_0)$ of differential forms at the point $x_0 \in X$ had dimension 2^n. Since $\Lambda(X) \simeq H(A)$, we may use $H(A)$ to define dim X indeed $\Lambda^p(X) = 0$ if $p > n$. As in 3.5, if A is noncommutative, we define the Hochshild dimension of A as the integer n such that $H^p(A) = 0$ if $p > n$.

2) If A is an algebra, then the set of $m \times m$ matrices with elements in A is also an algebra (all the algebra we are dealing with are supposed to be associative) and it is natural to ask what the Hochshild cohomology of $M_n(A)$ is. It can be proven that it is just the same. This suggests the definition of «Morita-equivalence»: two algebras A and B will be equivalent if $A \otimes K \simeq B \otimes K$ where K is the algebra of compact operators on a Hilbert space. $A \otimes K$ can be thought of as a space of matrices of any size with coefficients in A.

3) The previous remark tells us in particular that if A is a complex Clifford algebra associated with a nondegenerate scalar product, then $H_*(A) = \mathbb{C}$ if $* = 0$ and 0 if $* \geq 1$. Indeed, we know that A is isomorphic to a matrix algebra over $\mathbb{C}$ (or over $\mathbb{C} + \mathbb{C}$) , then, by Morita-equivalence $H_*(A) = H_*(\mathbb{C})$ and we use the results of Section 3.6.

4) Notice that if $f, g, h \in A$, and $f' \doteq f + (gh - hg)$, then, $bf' = bf =$ since $gh - hg = b(g\delta h)$ by eq. (13). Therefore, one always has $H_0(A) = A/[A, A]$ where A is the subspace of A spanned by all commutators in A (graded commutators if we are in the Z_2-graded case).

5) In the case where the algebra Der A of derivations of A is not zero, we have defined in 2.9 a space $\Lambda(\mathrm{Der}\, A, A)$ of exterior differential forms. The reader is invited to notice the differences between its definition and the construction of $H(A)$; both notions are intimately related when $A = C^\infty(X)$.

6) Notice that we do not know *yet* what is the noncommutative analog of the De Rham boundary operator ∂ on currents (or, from the dual point of view, the analog of the De Rham coboundary operator d on forms). In other words we do not know yet how to define De Rham Cohomology in a noncommutative set-up. The answer is Cyclic Cohomology and is the subject of the next section. The non commutative analogue of ∂ will be called B.

4. CYCLIC COHOMOLOGY

When $A = C^\infty(X)$, De Rham cocycles are in particular (equivalence classes of) differential forms ω and since $H(X) \sim \Lambda(X)$ it is therefore natural, in the noncommutative framework to try to define cyclic cohomology as a subcomplex of the Hochshild complex (or, at the homological level, as a quotient of the Hochshild complex). Actually, there are other equivalent definitions that will be given below.

4.1. The cyclic subcomplex of $H^*(A)$

Notice that the cyclic group of order $n+1$ acts on the space of monomials of the kind $a_0 \otimes a_1 \ldots \otimes a_n$ (cf. construction No. 2 of $\tilde{\Omega}(A)$ in Section 2.3) via $a_0 \otimes a_1 \ldots \otimes a_n \rightarrow a_n \otimes a_0 \ldots \otimes a_{n-1}$. Following [7], let us define the cyclicity operator λ as $(-1)^n$ times the generator of the cyclic group (4) acting on $\tilde{\Omega}(A)$ *i.e.*,

$$\lambda(a_0 \delta a_1 \ldots \delta a_n) \doteq (-1)^n a_n \delta a_0 \delta a_1 \ldots \delta a_{n-1} \tag{18}$$

and, more generally

$$\lambda(\omega \delta x) \doteq (-1)^{1+\partial\omega} x \delta\omega, \quad x \in A, \qquad \omega \in \tilde{\Omega}(A). \tag{18'}$$

At the dual level, the operator λ acts on forms φ on $\tilde{\Omega}(A)$ as folows:

$$[\lambda\varphi](\omega) \doteq \varphi(\lambda\omega). \tag{19}$$

When A is Z_2 graded, and if we express φ as a multilinear form, the same operator λ acting on φ can be written as follows

$$[\lambda\varphi](a_0, a_1, \ldots, a_n) = (-1)^n (-1)^{\partial a_n \sum_{i=0}^{n-1} \partial a_i} \varphi(a_n, a_0, \ldots, a_{n-1})$$

Notice that (18') implies $\lambda(\delta a_1 \ldots \delta a_n) = (-1)^n a_n \delta^2 a_1 \delta a_2 \ldots = 0$. The form φ is said to be cyclic if $\varphi o \lambda = \varphi$. Notice that if φ is cyclic, it vanishes over elements of the kind $\delta a_1 \ldots \delta a_n$, it is therefore automatically a Hochshild cochain (of Section 3.5 and eq. (14)). It is therefore natural to study the restriction of the Hochshild coboundary operator b to the space of cyclic forms. The main observation is that if φ is cyclic, so is $b\varphi$ (use eqs. (13), (14) and (19)): the set of cyclic forms is a subcomplex of the Hochshild complex and its cohomology is called cyclic cohomology and denoted $H_\lambda^*(A)$ (or sometimes $HC^*(A)$). Being a subcomplex, it may be that H_λ^* is not trivial even if H^* is. The graded commutator in the Z-graded algebra $\tilde{\Omega}(A)$ is defined as follows: $[\omega_1, \omega_2]_g = \omega_1\omega_2 - (-1)^{\partial\omega_1 \partial\omega_2} \omega_2\omega_1$. Notice that if φ is a cyclic cocycle it vanishes on $\delta\Omega(A)$. Now, from the definition of b (and β), we see that it vanishes on graded commutators of the type $[\omega, x]_g$, $x \in A$, but also, using cyclicity, it vanishes on graded commutators of the type $[\omega, \delta x]_g$, and therefore on all graded commutators. It is not too difficult to show that these properties characterize the cyclic cocycles. In other words, we could have defined the cyclic cocycles of degree n as graded traces of dimension n on the algebra $\tilde{\Omega}(A)$ that vanish over $\delta\tilde{\Omega}(A)$. (A graded trace being by definition a form vanishing over graded commutators.)

(4) The reader should be warned that, in ref. [7], the symbol λ denotes the cyclic permutation, without the sign $(-1)^n$.

4.2. Cyclic homology ♠

Of course it is possible to define a dual theory at the homological level; however, since the cyclic complex is a subcomplex of the Hochshild complex at the cohomological level, it will appear as a quotient at the homological level (5). Specifically, the cyclic chains will appear, as elements of $A^{\otimes n+1}/(1-\lambda)A^{\otimes n+1}$. In the next sections, we will stay at the cohomological level rather than homological because the calculations are usually simpler. Before ending this subsection, let us mention that whenever one can define cyclic objects in a category [18], *i.e.*, whenever we have a simplicial complex X_n (with face and degeneracy maps) and an extra structure given by an action of the cyclic group of order $n+1$ on X_n, one can define not only the Hochshild homology of the complex but also its cyclic homology [18, 14, 19]. This general technique leads to $H_{\lambda *}(A)$ when the cyclic object (called A in [18]) is built out of an algebra A by taking $X_n \doteq A^{\otimes n+1}$.

4.3. Cyclic cohomology and the Cuntz algebra $Q(A)$ ♠

In Section 2.8 we introduced the Cuntz and Zekri algebras. $Q(A)$ and $\epsilon(A)$. We will indicate now how they can be used to describe cyclic cohomology. Indeed let T be a trace on $Q(A)$, then define $\varphi(\omega) = T(\delta\omega)$ if $\omega \in \Omega(A)$ is even and $\varphi(\omega) = 0$ if ω is odd. Let us show that φ is an (even) cyclic cocycle on A [20]. Assume ω and ω' even, then

$$
\begin{aligned}
\varphi(\omega\omega') &= T(\delta(\omega\omega')) = T(q(\omega\omega')) = \\
&= T(q\omega.\omega' + \omega.q\omega' - q\omega.q\omega') = \\
&= T(\omega'.q\omega + q\omega'.\omega - q\omega.q\omega') = \\
&= T(\omega'\delta\omega + (\delta\omega')\omega)) = T(\delta(\omega'\omega) = \\
&= \varphi(\omega'\omega).
\end{aligned}
$$

One has still to consider the case ω, ω' odd (one finds the same result) and the mixed case (then $\varphi(\omega\omega') = 0$). Besides, the fact that φ is closed for δ is obvious. Therefore φ is indeed an (even)-graded trace on $\Omega(A)$ vanishing on $\delta\Omega(A)$, hence a cyclic cocycle. One can show [23] that the even cyclic cohomology can be reconstructed from the study of traces on QA . It is nice to remember that things are sometimes nicer in

(5) Notice that if we had started with a bigger Hochshild complex (cf. Remark at the end of 3.5) without assuming $\varphi o\delta = 0$ and defined therefore subsequently a reduced Hochshild cohomology, there would be no need to define a reduced cyclic cohomology because of the property φ cyclic $\rightarrow \varphi o\delta = 0$. The structure is quite the opposite at the homological level (where there is no need to define a reduced Hochshild homology but where some authors introduce a restricted cyclic homology [11]).

terms of $(Q(A), q)$ than in terms of $(\Omega(A), \delta)$: in the present situation, we do not have to assume $\delta\varphi = 0$ and we replace graded traces by traces. The relation between ϵA and the odd-dimensional cyclic cohomology is similar and is studied in [23]. Notice that the above establishes a relation between n-cocycles and traces on the ideal of $Q(A)$ spanned by elements $x^0 qx^1 ... qx^n$ and $qx^1 ... qx^n$; therefore one could be tempted of studying traces on the whole of $Q(A)$, not only on a particular ideal, such traces would define cocycle for each (even) n. Pursuing this idea leads to the definition of «entire cyclic cohomology» and we will come to it in Section 7.

4.4. Cyclic cohomology via mixed complexes ♠

Let (M, b) be a cochain complex (with $b^2 = 0$, b of degree +1) and (M, B) a chain complex (with $B^2 = 0$, B of degree -1); moreover, we assume that $bB + Bb = 0$. Such an object (M, b, B) is called a mixed complex. To each mixed complex, we may associate a chain complex $({}^B M, \Delta)$ as follows:

$$
\begin{aligned}
{}^B M^n &\doteq M^n \oplus M^{n-2} \oplus M^{n-4} \oplus \dots \\
\Delta &= b + B.
\end{aligned}
$$

It is clear from the above that $\Delta^2 = 0$ and that Δ maps ${}^B M^n$ into ${}^B M^{n+1}$. By definition, the cyclic cohomology of this mixed complex is the cohomology of $({}^B M, \Delta)$; [11, 21]. In order to justify the terminology, one has to show how such a mixed complex arises naturally in the universal differential envelope of an algebra A and to prove that the cyclic cohomology defined here coincides with the one defined in 4.1. More details will be given in 5.3.

4.5. The operators B_0 and B

The non-antisymmetrized boundary operator B_0

Let A be a unital algebra with unit and $\Omega(A)$ its differential envelope (if A is not unital, we add a unit). Let φ be a normalized (6) Hochshild cochain in degree $n+1$; it can be considered as a linear form on $\Omega(A)$ which can be written as $\varphi(a^0, a^1, \dots, a^{n+1}) = \varphi(a^0 \delta a^1 \dots \delta a^{n+1}), a^i \neq z, i > 0, z \in C$. Then $B_0\varphi$ is the n-cochain defined as follows:

$$
(B_0\varphi)(a^0, \dots, a^n) = \varphi(1, a^0, \dots, a^n). \tag{20}
$$

(6) We remind the reader that a normalized Hochshild cochain is a multilinear form on the space of $(a_0, a_1, \dots, a_n)$ which vanishes whenever $a_i, i > 0$ belongs to the image of the algebra of complex numbers in A .

More generally, if ω denotes an arbitrary element in $\Omega(A)$, we may define B_0 as

$$[B_0\varphi](\omega) = \varphi(\delta\omega). \tag{21}$$

The cyclicly-antisymmetrized boundary operator B

Let us first introduce the operator A of cyclic antisymmetrization which is defined as (7)

$$A = 1 + \lambda + \lambda^2 + \ldots + \lambda^n \qquad \text{on} \qquad \widetilde{\Omega}_n(A) \tag{22}$$

with λ as in eq. (18). At the dual devel, let us also call A the operator $(A\varphi)(\omega) \doteq \varphi(A\omega)$. The coboundary operator B is then defined as $\mathsf{B} = \delta A$ in $\widetilde{\Omega}(A)$ *i.e.*,

$$\mathsf{B}(a_0\delta a_1 \ldots \delta a_n) = \sum_{i=1}^{n} (-1)^{jn}(\delta a_j \delta a_{j+1} \ldots \delta a_n \delta a_0 \ldots \delta a_{j-1}) \tag{23}$$

or, at the dual level, as $B = AB_0$, *i.e.*,

$$(B\varphi)(\omega) = \varphi(\mathsf{B}\omega). \tag{24}$$

It is now just a matter of algebraic manipulations to show that $B^2 = 0$ and that $Bb = -bB$. The reader should for instance compare $\mathsf{B}b(\omega)$ and $b\mathsf{B}(\omega)$, with $\omega = a_0\delta a_1\delta a_2$. Notice that B and B_0 are operators of degree -1. We are therefore in the situation of Section 4.4: calling C^n the space of Hochshild cochains in dimension n (*i.e.*, the space of forms $\varphi(a^0, \ldots, a^n)$). We see that (C, b, B) is a mixed complex. The definition of the operators B_0 and B in the unnormalized Hochshild complex is slightly more involved. One has to set

$$(B_0\varphi)(a^0, \ldots, a^n) = \varphi(1, a^0, \ldots, a^n) - (-1)^{n+1}\varphi(a^0, \ldots, a^n, 1)$$

Then one sets $B = AB_0$ as before. But, since only normalized Hochshild cocycles (where 1 can only appear in first position) have a nice interpretation in terms of the differential algebra $\Omega(A)$, we will not use this.

Before ending this paragraph let us mention one useful property of the operator A which is a direct consequence of its definition (eq. (22)): If τ is a cyclic n-cocycle, then $A\tau = (n+1)\tau$. We will return to the study of the properties of the operator B in 5.5 but meanwhile, we are ready for studying a few examples.

---

(7) The definition of λ involves a sign, cf. footnote 4 in sect. 4.1

4.6. Examples

4.6.1. Cyclic cohomology of the algebra of complex numbers

This is a continuation of the examples in 2.7 and 3.6. The cyclicity condition $\varphi_0 \lambda = \varphi$ imposes $\varphi(q^{2n+1}) = \varphi(eq^{2n+1}) = 0$ and $\varphi(q^{2n}) = 0$. Therefore, in the odd case there are no nontrivial cyclic cochains:

$$C_\lambda^{2n+1} = Z_\lambda^{2n+1} = B_\lambda^{2n+1} = H_\lambda^{2n+1} = 0.$$

In the even case, we get $C_\lambda^{2n} = \mathrm{C}$ since $\varphi(eq^{2n})$ is not determined. The condition $b\varphi = 0$ does not bring anything new in this case since

$$b\varphi(q^{2n+1}) = \varphi(\beta q^{2n+1}) = 0$$

and

$$b\varphi(eq^{2n+1}) = \varphi(\beta eq^{2n+1}) = 0.$$

Therefore all cyclic cochains are cocycles $C_\lambda^{2n} = Z_\lambda^{2n}(= \mathrm{C})$. Moreover, if $\psi \in C_\lambda^{2n-1}$, ψ is zero (and $b\psi = 0$). Therefore $B_\lambda^{2n} = 0$ and $H_\lambda^{2n} = \mathrm{C}$. The cyclic cohomology of complex numbers is periodic modulo 2. Notice that Hochshild cohomology groups are essentially trivial (but in dimension 0), but cyclic cohomology groups are not.

4.6.2. Cyclic cohomology of the commutative algebra $A = C^\infty(X)$

This is a continuation of the example 3.7. Let us call cl (standing for «classical») the universal morphism from $\Omega(A)$ to $\Lambda(X)$. It is instructive to consider the following cases:

$$\mathrm{B}(a_0\delta a_1\delta a_2) = \delta a_0\delta a_1\delta a_2 + \delta a_1\delta a_2\delta a_0 + \delta a_2\delta a_0\delta a_1$$

$$cl(\mathrm{B}(a_0\delta a_1\delta a_2) = 3\,da_0 \wedge da_1 \wedge da_2$$

and

$$\mathrm{B}(a_0\delta a_1\delta a_2\delta a_3) = \delta a_0\delta a_1\delta a_2\delta a_3 - \delta a_1\delta a_2\delta a_3\delta a_0 + \\ + \delta a_2\delta a_3\delta a_0\delta a_1 - \delta a_3\delta a_0\delta a_1\delta a_2$$

$$cl(\mathrm{B}(a_0\delta a_1\delta a_1\delta a_3)) = 4\,da_0 \wedge da_1 \wedge da_2 \wedge da_3.$$

It is clear that B appears as the generalization of the De Rham coboundary operator (actually B_0 is enough in this case because of the antisymmetry of $\wedge$).

At the dual level, cyclic cohomology will appear here as the De Rham homology of currents. Let C be a k-dimensional current and σ a ($k-1$)-form, then $\langle C, d\sigma\rangle = \langle \partial C, \sigma\rangle$ where ∂ is the De Rham boundary for currents. Let φ the Hochshild cocycle corresponding to C (considered as a graded trace of dimension k on $\Omega(A)$, i.e.

$$\varphi(f^0\delta f^1\ldots\delta f^k) = \langle C, f^0 df^1 \wedge\ldots\wedge df^k\rangle \tag{25}$$

Then,

$$\begin{aligned}\langle \partial C, f^0\delta f^1\ldots\delta f^{k-1}\rangle &= \langle C, df^0 df^1 \wedge\ldots\wedge df^{k-1}\rangle = \\ &= (B_0\varphi)(f^0\delta f^1\ldots\delta f^k)\end{aligned} \tag{26}$$

However, the correspondance between cyclic cohomology *of degree k* and De Rham homology *of degree k* is not one to one. Indeed, let φ_1 be a k-dimensional cyclic cocycle, it determines a De Rham current C (by eq. 16) which is closed. But, in turn, C determines a cyclic cocycle φ by eq. 25. The problem is that the class of $\varphi_1-\varphi$ is zero in H^k but not in H^k_λ. In other words, although $\varphi_1-\varphi$ is a non trivial k-dimensional cyclic cocycle, its image under « Cl » (the «classical» homomorphism) is trivial. One can show that, in this case, there exists $\psi \in H^{k-2}_\lambda$ and an operator S, from H^{k-2}_λ to H^k_λ such that $(\varphi_1-\varphi) = S\varphi$. We will return to the definition of S in the next section. However, ψ is not determined uniquely and, again, one can find $\psi_1 \in H^{k-2}_\lambda$ such that $S\psi_1 = S\psi$ with $\psi_1 - \psi = S\eta$ for some $\eta \in H^{k-4}_\lambda$. etc. The result is that for each k, H^k_λ is isomorphic to $Ker\partial$ (in the space of k-dimensional currents) $\oplus H_{k-2}(X) \oplus H_{k-4}(X) \oplus \ldots$ where H_k denotes the De Rham homology of X.

4.6.3. Z_2-graded cyclic cohomology of Grassman algebras

Let us consider for example the algebra $\wedge C^2$ generated over the complex numbers by the elements $1, a, b,$ with the relations $a^2 = b^2 = 0$, $ab = -ba$. This algebra is clearly Z_2-graded (we give an intrinsic grade 1 to the odd generators a and b). We will therefore consider the Z_2-graded cyclic cohomology of this algebra (super-cyclic cohomology!) –cf. sect. 2.5 and 3.8.

At the lowest level, a zero cochain φ should satisfy the relation $b\varphi(a_0, a_1) = 0$, but, by definition of b, this is nothing else that the graded commutator of a_0 and a_1. This graded commutator always vanishes since the algebra is graded commutative. Therefore, H^0_λ is generated by the classes of the linear forms $\varphi_1, \varphi_a, \varphi_b, \varphi_a b$ where $\varphi_x(y) = 1$ if $x = y$ and 0 in the other cases. Notice that $\varphi_a b$ is the Berezin integral (defined up to scale) associated with this set of generators. We can write $H^0_\lambda = C^2 \widehat{\oplus} C^2$. It is convenient to single out the cohomology of the subalgebra of complex numbers (generated by the element 1) and write $H^0_\lambda = C \oplus C\widehat{\oplus}C^2$. At the next order, we observe that cyclicity ($\varphi\lambda = \varphi$) imposes the following constraints. $\varphi(1,1) = -\varphi(1,1), \varphi(1,a) = -\varphi(a,1), \varphi(1,b) = -\varphi(b,1), \varphi(1,ab) = -\varphi(ab,1), \varphi(a,b) = \varphi(b,a), \varphi(a,ab) = -\varphi(ab,a), \varphi(ab,ab) = -\varphi(ab,ab)$. This implies in particular that $\varphi(1,1) = \varphi(ab,ab) = 0$. Also the Hochshild condition $b\varphi = 0$ implies $\varphi(1,ab) = 0$; this comes from $[b\varphi](1,a,b) = 0$, indeed $[b\varphi](1,a,b) = \varphi(a,b) - \varphi(1,ab) - \varphi(b,a) = -\varphi(1,ab)$. The Hochshild condition also implies $\varphi(1,a) = 0$, this comes from $[b\varphi](1,1,a) = 0$. In the same way $\varphi(1,b) = 0$. The space of cyclic cocycles is therefore generated by three even cocycles φ_1 , φ_2 , φ_3 and two odd cocycles φ_4 and φ_5 which do not vanish only on the following arguments: $\varphi_1(a,a), \varphi_2(b,b), \varphi_3(a,b), \varphi_4(a,ab) = -\varphi_4(ab,a), \varphi_5(b,ba) = -\varphi_5(ba,b)$. Since, from the other hand the space of 1-cyclic coboundaries is clearly zero, we get $H^1_\lambda = C^3\widehat{\oplus}C^2$. One could compute in the same way H^2_λ, H^3_λ etc. Actually, there is a shorter way which uses the result of the Z_2 graded cyclic cohomology of ΛC along with a Kunneth formula [33]. Before stating the general result, let us notice that, the complex H^*_λ being Z_2-graded, it is convenient to introduce a Z_2-graded Poincaré polynomial. When V^* is a Z-graded and Z_2-graded vector space, one defines the following polynomial. $P(V^*)(t) = \Sigma_n(dim(V^n)^+ + \theta dim(V^n)^-)t^n$ where θ is the generator of $Z_2(\theta^2 = 1$. The general result for the Grassman algebra ΛC^r is $H^*_\lambda(\Lambda C^r) = H^*_\lambda(C) + V^*$ where the first term denotes the cyclic cohomology of complex numbers (cf. section 4.6.1) and where the second term is a graded (and Z_2-graded) vector space whose Poincaré polynomial is

$$P(t) = [2^{r-1}(1+\theta) - (1-t)^r]/[(1+t)(1-t)^r]$$

One finds, [33], that in the case of ΛC ,

$$P(t) = (\theta + t)/(1-t^2) = \theta + t + \theta t^2 + t^3 + \theta t^4 + t^5 + \dots$$

In the particular case of ΛC^2 , we get $P(t) = (1+2\theta) + (3+2\theta)t + (3+4\theta)t^2 + (5+8\theta)t^3 + \dots$, in accordance with our previous explicit calculations (notice that, at order 2, the cohomology will be $H^2_\lambda = C \oplus C^3\widehat{\oplus}C^4$

The Z_2-graded cyclic cohomology of Clifford algebras $G(n)$ is actually much simpler. One proves [33] that H^*_λ is generated by $\tau, S\tau, S^2\tau, S^4\tau, \dots$ where τ is the graded trace determined by $\tau(\gamma_1\gamma_2\dots\gamma_n) = 1$ and τ (other generators) $= 0$. For instance, in the case of the algebra $G(4)$ generated by the symbols $\gamma_i, i = 1\dots 4$, with the relations $\gamma_i^2 = 1, \gamma_i\gamma_j + \gamma_j\gamma_i = 0, i \neq j$, one can check that $Z^0_\lambda = C$ is generated by τ with $\tau(1) = \tau(\gamma_i) = \tau(\gamma_i\gamma_j) = \tau(\gamma_i\gamma_j\gamma_k) = 0$ and $\tau(\gamma_1\gamma_2\gamma_3\gamma_4) = 1$. At the next order $Z^1_\lambda = 0$. Then $Z^2_\lambda = C$ is generated by $\varphi = 1/(2i\pi)S\tau$ with $\varphi(a,b,c) = (-1)^{\partial b}$ if $abc = \pm\gamma_1\gamma_2\gamma_3\gamma_4$ and 0 in the other cases. Notice that if we set $\Gamma_1 = \gamma_1, \Gamma_2 = \gamma_2$ and $\bar{\Gamma}_1 = \gamma_3, \bar{\Gamma}_2 = \gamma_4$ then $\eta = 1/2(\Gamma_1 + i\Gamma_2)$ and $\bar{\eta} = 1/2(\bar{\Gamma}_1 + i\bar{\Gamma}_2)$ generate the Grassman algebra $\wedge C^2$. It is easy to check that $\varphi(\eta, \eta, \bar{\eta}) = 0, \varphi(\bar{\eta}, \eta, \bar{\eta}) = 0, \varphi(\lambda, \eta, \bar{\eta}) = 0$ if $\lambda \in C$, but $\varphi(\eta\bar{\eta}, \eta, \bar{\eta}) \neq 0$ since the product $\eta\bar{\eta}\eta\bar{\eta}$ contains a term of the kind $\gamma_1\gamma_2\gamma_3\gamma_4$. This could be used as another way of studying the properties of the Berezin integral.

5. CYCLIC COHOMOLOGY, PERIODICITY AND COBORDISM IN NON-COMMUTATIVE GEOMETRY

5.1. The periodicity operator S

In the rest of this paper we use cohomology rather than homology but it should be understood that there is an analogous theory at the dual level. Let A and B be two algebra. Then, in general $\Omega(A \otimes B) \neq \Omega(A) \otimes \Omega(B)$ but we get a natural homomorphism π from the first into the second because of the universal property of $\Omega(A \otimes B)$ of Section 2. Let $\varphi \in C^n(A)$ and $\psi \in C^m(B)$ be two Hochshild cochain, they can be thought of as linear forms $\hat{\varphi}$ and $\hat{\psi}$ on $\Omega(A)$ and $\Omega(B)$. Their cup product $(\varphi \# \psi)$ is a Hochshild cochain (of degree $n+m$) defined by $(\varphi \# \psi)\hat{} = (\hat{\varphi} \otimes \hat{\psi})\pi$. These formal considerations get simplified if we take $B = \mathbb{C}$ (the complex numbers) and if we take for ψ the 2-cocycle τ which generates the cyclic cohomology of the algebra $\mathbb{C}$ (cf. Section 4.6.1); we can choose $\tau(e,e,e)(= \hat{\tau}(e\delta e\delta e)) = 2i\pi$. Remember that we call C^n the Hochshild cochains and C^n_λ the cyclic cochains. Then $\varphi \# \tau \in C^{n+2}(A \otimes C) = C^{n+2}(A)$ when A is a complex algebra; we therefore get a map from $C^n(A)$ into $C^{n+2}(A)$. One can prove that if φ and ψ are Hochshild cocycles, then $\varphi \# \psi$ is still a Hochshild cocycle. Actually, the map of interest, called S (cf. [7]), is gotten by restricting our attention to the cyclic subcomplex C^*_λ. Let $\varphi \in C^n_\lambda(A)$, then $S\varphi$ is defined as:

$$S\varphi = \frac{1}{n+3} A(\varphi \# \tau) \tag{28}$$

where A is the cyclic symmetrizer introduced in eq. 22.

Facts about the operator S (cf. [7]):

(i) S maps C^n_λ into C^{n+2}_λ; (in particular $S\varphi$ is still cyclic).

(ii) If φ is a cyclic cocycle (*i.e.* $\varphi \in Z^n_\lambda$, *i.e.* $b\varphi = 0$) then $S\varphi \in Z^{n+2}_\lambda$ and $S\varphi = \varphi \# \tau$ (this explains why we introduced the coefficient $1/(n+3)$ and the operator A in the general definition of S): the image under S of a cyclic cocycle is a cyclic cocycle.

(iii) If φ is a cyclic coboundary (*i.e.* $\varphi \in B^n_\lambda$, *i.e.* $\varphi = b\psi$ for $\psi \in C^{n-1}_\lambda$) then $S\varphi$ is also a cyclic coboundary: $S\varphi \in B^{n+2}_\lambda$. More precisely

$$bS\psi = \frac{n+1}{n+3} Sb\psi \quad \text{for} \quad \psi \in C^n_\lambda. \tag{29}$$

(iv) From the above we have that S induces a map (also called S) from the cyclic cohomology groups $H_\lambda^n \to H_\lambda^{n+2}$.

(v) If we represent cyclic cochains as linear forms $\hat{\varphi}$ on $\Omega(A)$, we have the following explicit writing for $S\hat{\varphi}$: let $\omega \in \Omega_n(A)$, then

$$(S\hat{\varphi})(\omega) = \frac{2i\pi}{n+3}\,\hat{\varphi}(SA\omega)$$

where A is defined in eq. 22 and S is defined as follows

$$\left.\begin{aligned} S(a_0\delta a_1\ldots\delta a_{n+2}) &= a_0 a_1 a_2 \delta a_3 \ldots \delta a_{n+2} + \\ &\quad + \sum_{i=2}^{n+1}(a_0\delta a_1\ldots\delta a_{i-1})(a_i a_{i+1})(\delta a_{i+2}\ldots\delta a_{n+2}) \\ S(a_0) = S(a_0\delta a_1) &= 0 \end{aligned}\right\} . \tag{31}$$

This is actually an alternative for the abstract definition eq 28. If φ is a cyclic cocycle ($b\varphi = 0$), we may remove both $1/(n+3)$ and the operator A in the above formula.

(vi) We saw in (ii) that if $\varphi \in Z_\lambda^n$ then $S\varphi \in Z_\lambda^{n+2}$ but moreover we can show that $S\varphi$ is also a Hochshild coboundary: $S\varphi = b\psi$ (not a cyclic coboundary since ψ is usually not cyclic). This property can be checked by taking

$$\psi(a^0,\ldots,a^{n+1}) = 2i\pi\sum_{j=1}^{n}(-1)^j\hat{\varphi}((a^0\delta a^1\ldots\delta a^{j-1})a^j(\delta a^{j+1}\ldots\delta a^n)). \tag{32}$$

The operator S is sometimes called the periodicity operator or the suspension operator.

5.2. Relation between the operators b, B and S ♠

The following results come from algebraic manipulations involving the definitions of b, B, B_0 and S ; we refer to [7] for the details and the proofs.

(i) The image under B of a Hochshild cochain is always cyclic moreover the map is onto (but not one to one): $BC^n = C_\lambda^{n-1}$.

(ii) The image under B of a Hochshild cocycle is a cyclic cocycle (it is cyclic because of (i) and a cocycle since $Bb = -bB$). Moreover it also lies in the image of B_0 (the precise relation is $BZ^n = B_0 Z^n \cap Z_\lambda^{n-1}$).

Notice that a given cyclic cocycle τ can a priori be written $\tau = B\varphi$, where $\varphi \in C^n$ because of (i); however if $\tau \notin B_0 Z^n$ it cannot be written as $B\varphi$ with $\varphi \in Z^n \subset C^n$ because of (ii). If we start with a given cyclic cocycle $\tau \in Z_\lambda^{n-1}$, we may distinguish 2 cases 1) $\tau \in B_0 Z^n$, then, because of (ii) it can be written as $\tau = B\varphi$ for some $\varphi \in Z^n$. Then $\omega = b\varphi = 0$ and defines the same cohomology class (zero!) as

$S\tau \in B_\lambda^{n+1} \subset Z_\lambda^{n+1}$. 2) $\tau \notin B_0 Z^n$, then, because of (i) it can be written as $\tau = B\varphi$ for some $\varphi \in C^n$ (actually it can also be written $\tau = B_0\psi$ for some ψ in C^n). Then $\omega = b\varphi \in B^{n+1} \subset Z^{n+1}$ and we have 2 subcases. 2a) Case where ψ is a cyclic cocycle: $\omega \in Z^\lambda_{n+1}$ (it has no reason to be a cyclic coboundary since φ is not cyclic a priori). In this case one can show that ω defines (up to a scalar factor), the same cohomology class as $S\tau$; more precisely $[SB\varphi] = 2i\pi n(n+1)[b\varphi]$. One could even choose an element φ_0 in C^n such that the identify holds at the cocycle level. 2b) Case where ω is not cyclic: $\omega \notin Z_\lambda^{n+1}$. Then, in any case $\omega \in \ker b \cap \ker B$ ($b\omega = b^2\varphi = 0$ and $B\omega = Bb\varphi = b\tau = 0$) and, in this situation, it is possible to «correct» it: one can find canonicaly $\widetilde{\omega}$ in the same Hochshild cohomology class as ω such that $\widetilde{\omega}$ is cyclic [7,II p.121]; then, again, $[S\tau] = 2i\pi n(n+1)[\widetilde{\omega}]$ in H_λ^{n+1} . More precisely, one proves that $\ker b \cap \ker B = Z_\lambda + b(\ker B)$, therefore one writes $\omega = \widetilde{\omega} + b\psi$ where ψ , determined by the equation $(1-\lambda)\psi = B_0\omega$, belongs to $KerB + Z^n \subset C^n$ then $b\psi \in b(KerB)$. This last remark will be used in Section 6.

The following diagram may help to remember the above relations (in case 2a).

$$\begin{tikzcd}
Z_\lambda^{n+1} & & \omega \arrow[rrr, phantom, "\doteq"] & & & S\tau \\
Z^n & \varphi \arrow[ur, "b"] \arrow[dr, "B"'] & & & & \\
Z_\lambda^{n-1} & & \tau \arrow[uurrr, "S"] & & &
\end{tikzcd}$$

The previous results suggest that, at the cohomological level, one may write

$$S : H_\lambda^{n-1} \to H_\lambda^{n+1} \quad \text{as} \quad S = 2i\pi n(n+1)bB^{-1}.$$

This is indeed true and will be precised in the following section.

5.3. The Connes sequence

Since B maps Hochshild cochains onto cyclic cochains, it also induces a map at the level of cohomology groups: B maps H^* into H_λ^* . Also, the operator S maps H_λ^* into H_λ^* (actually H_λ^n into H_λ^{n+2}). Finally, a cyclic cocycle is in particular a Hochshild cocycle and we have therefore a map I (inclusion) from H_λ^* into H^* . It is tempting to consider the triangle

$$\begin{tikzcd}
H_\lambda^* \arrow[rr, "S"] & & H_\lambda^* \arrow[dl, "I"] \\
& H^* \arrow[ul, "B"] &
\end{tikzcd}$$

The nice thing is that this triangle is exact (the image of any of the three groups appearing in this triangle under the corresponding map is the kernel of the following map), in other words, we have an exact couple (for elementary properties of exact couples, cf. [22]). The simplicity of this result relating Hochshild to cyclic cohomology should not make the reader think that the proof is simple: it is probably one of the most difficult technical points of the theory. The result encodes some of the properties already mentioned in 5.2; notice that it implies that $SB = IS = BI = 0$.

Since $H^* = \oplus_n H^n$ and $H^*_\lambda = \oplus_n H^n_\lambda$, and if we remember that S increases n by 2 and B decreases n by one, we may restate the above result by saying that we have a infinite exact sequence:

$$\ldots \to H^n \xrightarrow{B} H^{n-1}_\lambda \xrightarrow{S} H^{n+1}_\lambda \xrightarrow{I} H^{n+1} \xrightarrow{B} H^n_\lambda \to \ldots.$$

Notice that if $n >$ (Hochshild dimension of A –cf. 3.6.), $H^n = 0$, then, we have:

$$0 \to H^{n-1}_\lambda \xrightarrow{S} H^n_\lambda \to 0;$$

this shows that H^{n-1}_λ and H^n_λ are isomorphic (periodicity modulo 2) under S .

The above result justifies the following definition: We define the periodic cyclic cohomology of the algebra A (groups denoted by H^*_{per}) as the inductive limit of the group H_λ under the maps $S : H^n_\lambda \to H^{n+2}_\lambda$. These groups were actually «De Rham cohomology groups» in [7], [8] but this terminology is slightly confusing and anyway, the new terminology, as well as the notation H^*_{per} seems to become standard.

Since we have an exact couple, it is clear that $IB : H^* \to H^*$ is such that $(IB)^2 = I(BI)B = 0$, therefore one gets in this way a derived couple

$$\begin{array}{ccc} A_2 & \longrightarrow & A_2 \\ & \nwarrow \quad \swarrow & \\ & E_2 & \end{array}$$

where $A_2 = S(H^*_\lambda)$ and $E_2 = \text{Ker}\ (IB)/Im(IB)$: One can then build a sequence of derived couples (A_p, E_p). The corresponding spectral sequence $\{E_p\}$ whose first term is E_2 can be shown to be convergent. The reader unfamiliar with spectral sequences should skip the next subparagraph and just remember the definition of E_2 and the definition of H^*_{per} given above.

We already introduced mixed complexes in Section 4.4, it is clear that (C^*, b, B) where C^* denotes the space of Hochshild cochains is a mixed complex. We find a double complex associated with it as follows ([7]): We define $C^{n,m} \doteq C^{n-m}$, then, if $\varphi \in C^{n,m}$, we set

$$d_1\varphi = (n-m+1)b\varphi, \quad d_2\varphi = \frac{1}{(n-m)}B\varphi.$$

One can then consider two possible filtrations (and two spectral sequences); the initial term E_2 of the spectral sequence associated to the first filtration is zero and does not converge in general but the second filtration is convergent (towards the associated graded complex) and actually coincides with the spectral sequence associated with the exact couple. The cohomology of the double complex depends only upon the parity of n one finds $H^n(C^{**}, d_1 + d_2)$ equal to $H^{\text{even}}_{\text{per}}$ $(= \oplus_m H^{2m}_{\text{per}})$ if n is even or to $H^{\text{odd}}_{\text{per}} = \oplus_m H^{2m+1}_{\text{per}}$ if n is odd.

5.4. Cycles over an algebra and noncommutative cobordism

The rather formal developments of the preceeding sections should not cloud the fact that a cyclic cocycle is, roughly speaking, a noncommutative generalization of the symbol « $\int$ ». Indeed, if M is an n-dimensional manifold without boundary and $f_0, f_1, \dots f_n$ are $(n+1)$ functions on M (elements of $A = C(M)$), we may calculate the number

$$\int_M f_0\, df_1 \wedge df_2 \wedge \dots \wedge df_n .$$

The operator $\int_M$ appears as a cyclic cocycle for the M algebra A and, as discussed in Sections 4.1 and 4.6.2, it appears also as a graded trace on the universal differential algebra $\Omega(A)$ vanishing on $\delta\Omega(A)$ (so a closed graded trace for δ). It finally appears as a closed graded trace of the graded differential algebra $\Lambda(M) = \oplus_{p=0}^{n} \ \Lambda^p(M)$ of differential forms on M (this is not a surprise since $\Omega(A)$ is universal). This last property motivates the following generalization.

A cycle of dimension n over an associative algebra A is a graded differential algebra $\Lambda = \oplus_{p=0}^{n} \Lambda^p$ with differential d along with a homomorphism $A \xrightarrow{\rho} \Lambda^0$ and a closed graded trace $\int$ from Λ^n into $\mathbb{C}$ (the adjective closed referring here to the property $\int dw = 0$, $\forall \omega \in \Lambda^{n-1}$). With such a definition, it is clear that $(\Lambda(M), \int_M)$ is a cycle of dimension n over $A = C(M)$ when M is a smooth manifold without boundary of dimension n; more generally, in the case where M has non trivial homology one can construct other cycles.

Given an n-dimensional cycle, we shall define (as in [7]) its character by the following $(n+1)$-linear functional on A :

$$\tau(a^0, \dots, a^n) = \int \varphi(a^0)\, d\rho(a^1) \dots d\rho(a^n). \tag{32}$$

It is almost clear (and anyway true) that it is equivalent for a $(n+1)$-linear functional τ on A to be

(i) a cyclic cocyle,

(ii) a closed graded trace on $\Omega(A)$,

(iii) the character of a cycle $(\Lambda, \int)$ over A. As we shall see later, this last characterization of cyclic cocyles is one of the most «operational» since it allows us to build them explicitly.

In our previous analogy with «commutative geometry», we used a manifold without boundary since we wanted to write (using Stoke's theorem)

$$\int_M d\omega = \int_{\partial M} \omega = 0\,.$$

When $\partial M \neq 0$, we should consider *two* differential algebras, namely $\Lambda(M)$ and $\Lambda(\partial M)$ that we may call $\partial\Lambda(M)$. In the noncommutative set up, on [7] is therefore lead to the following definitions:

We first consider a triple $(\Lambda, \partial\Lambda, \int)$ where Λ and $\partial\Lambda$ are differential algebras of dimensions n and $n-1$ [and $\int$ a non closed graded trace on Λ]; in order to mimic what we have in the commutative case (a p-form on M defines also a possibly vanishing p-form on ∂M), we assume that we are given a subjective morphism r : $\Lambda \to \partial\Lambda$, in the present case, we cannot assume that $\int d\omega$ vanishes when $\omega \in \Lambda^{n-1}$ but we may impose that $\int d\omega = 0$ whenever $\omega \in \Lambda^{n-1}$ is such that $r(\omega) = 0$. Moreover we define a graded trace $\int'$ or $\partial\Lambda$ by $\int \omega' = \int d\omega$ for any $\omega \in \Lambda^{n-1}$ with $r(\omega) = \omega'$. Such a triple $(\Lambda, \partial\Lambda, \int)$ along with the morphism r will be called a chain; by the boundary of such a chain we will mean the cycle $(\partial\Lambda, \int')$.

We are now ready for the definition of noncommutative cobordism. Two cycles Λ_1, Λ_2 over the algebra A (with homomorphisms ρ_1, ρ_2) are cobordant if there exist a chain Λ with boundary $(\Lambda_1 \oplus \Lambda_2, \int_1 - \int_2)$ and a homomorphism $\rho : A \to \Lambda$ such that $r.\rho = (\rho_1, \rho_2)$. One could check that this is an equivalence relation. It is not too difficult to prove that if τ_1 and τ_2 are the characters of two such cobordant cycles over A, then $\tau_1 - \tau_2 = B_0\varphi$ where $\varphi(a^0, \ldots, a^{n+1}) = \int \rho(a^0) d\rho(a^1) \ldots d\rho(a^n)$. There is more: one can prove using 5.2 (ii), the following theorem: two cycles over A are cobordant if and only if their characters τ_1 and τ_2 are such that $\tau_1 - \tau_2 = B\psi$ where $\psi \in H^{n+1}(A)$. The group $M^*(A)$ of noncommutative cobordism classes over A is therefore equal to the vector space $H^*_\lambda/\mathrm{Im}\ B$. Another interpretation of this group will be given in the next section (§5.5).

Before ending this section, let us mention that, later, we will allow n to be infinite (infinite dimensional cycles), then yielding functional over $\Omega(A)$ – via their character – and related to entire cyclic cohomoogy (cf. Section 7, 9.7 and 10.5).

5.5. The cohomology of De Rham-Karoubi ♠

Denoting as usual by $(\Omega A, \delta)$ the universal differential envelope of A, we may notice that the space

$$[\Omega A, \Omega A]_n \doteq \sum_{p+q=n} [\Omega_p A, \Omega_q A]$$

is stable under δ and it is tempting to consider the cohomology of the complex $(\Lambda^*\Omega, \delta)$ where $\Lambda^n(\Omega A) \doteq \Omega_n A/[\Omega A, \Omega A]_n$: this is the cohomology of «De Rham-Karoubi» introduced in [6]. It is sometimes called noncommutative De Rham cohomology but this name should be avoided because too many different cohomologies defined in the context of associative algebras coincide with the usual De Rham cohomology in the particular case where $A = C(X)$). At the dual level, one can define the homology of De Rham-Karoubi as the homology of the complex $\Lambda_*\Omega A$ of (graded) traces on ΩA under the operator δ^t – the transposed of δ –. It is shown in [7] – see also [11] – that the homology of this complex coincides with the group $M^*(A)$ of cobordism classes over A introduced in the previous section.

5.6. When A is abelian: the complex of Kähler-De Rham ♠

When A is an abelian algebra, it is standard to consider the following complex (we discuss it here just to show that its definition requires commutativity and to warn the reader who could come across this complex that it comes from a different construction). Let us call $\Omega_k^1 A = \Lambda^1(\Omega A) = \Omega_1 A/[A, \Omega_1 A]$; notice that $\Omega_k^1 A$ is a A-module, then $\Omega_k A$ is defined as the exterior differential algebra of $\Omega_k^1 A$ (one may introduce an interior product as usual). The cohomology of $(\Omega_k A, \delta)$, when A is abelian is the cohomology of Kähler-De Rham. Notice, that, by universality of ΩA, we have a surjection $\Omega A \rightarrow \Omega_k A$ (which vanishes on $[\Omega A, \Omega A]$) we therefore get also a surjection from $\Lambda\Omega(A)$ onto $\Omega_k A$ which is actually an isomorphism in degree 0 and 1. It is an isomorphism in degree ≥ 2 if and only if $[\Omega A, \Omega A]$ is a bilateral ideal of ΩA [11].

6. POSITIVE COCYCLES ♠

We already mentioned in 4.3 that traces on the Cuntz algebra QA lead to cyclic cocycle: if $\varphi \in Z_\lambda^n$, n even, then there exist a trace T on QA such that $\varphi(a^0, \ldots, a^n) = T(qa^0 \ldots qa^n)$. More generally we could consider multilinear maps ω defined by

$$\omega_{n+1}(a^0, \ldots, a^n) = T(a^0 qa^1 \ldots qa^n)$$

A careful study would reveal that such maps ω are such that $b\omega = 0$, are not cyclic in general but are such that $B_0\omega$ is invariant under the action of the cyclic group. In the case where A has a $*$ operation (same thing for QA), it is natural to study positive traces T on QA i.e. $T(\omega^*\omega) \geq 0 \ \forall\omega)$: they will correspond to some special (noncyclic) cocycles that have been christened «positive cocycles» [23]. From the physical point of view, it is natural to be particularly interested in positive functionals (bearing in mind the probabilistic interpretation of quantum mechanics). From the mathematical point of view, if T is a positive trace, one can then build a scalar product

$\langle \alpha, \beta \rangle = T(\alpha\beta^*)$ and get an Hilbertian algebra, in which case one obtains many nice properties (in particular one can find a representation via the GNS construction). The corresponding «positive cocycles» on A have the following properties (this can be used as a definition): let n be an even integer and ω an $(n+1)$ linear form over A, then ω is a positive cocycle iff

(i) $b\omega = 0$,

(ii) $(1+\lambda)B_0\omega = 0$,

(iii) the following scalar product on $A^{\otimes p+1}$, $p = n/2$ is positive: $\langle a^0 \otimes \ldots \otimes a^p, b^0 \otimes \ldots \otimes b^p \rangle = \omega(b^{0*}a^0, a^1, \ldots, a^p, b^{p*}, \ldots, b^{1*})$. When n is odd this definition has to be generalized. Such a positive cocycle ω is a Hochshild cocycle by (i) but is not cyclic in general, however, it is possible to find a cyclic cocycle $\widetilde{\omega}$ of the same order (but no longer positive) in the same Hochshild cohomology class. Indeed $(1+\lambda)B_0\omega = 0 \Rightarrow AB_0\omega = 0 \Rightarrow B\omega = 0$ therefore $\omega \in \operatorname{Ker} b \cap \operatorname{Ker} B$ and we can use the method introduced in 5.2 2b): in the present case it is enough to choose $\psi \doteq \frac{1}{2} B_0\omega$ (then $(1-\lambda)\psi = B_0\omega$ as it should); in other words, if ω is a positive cocycle, then $\widetilde{\omega} = \omega - \frac{b}{2} B_0\omega$ is cyclic. Let us give two examples.

1) Let Σ be a Riemannian surface and f^0, f^1, f^2 three functions on Σ. Then the functional $\omega(f^0, f^1, f^2) = \frac{i}{2} \int f^0 \partial f^1 \bar{\partial} f^2$ is a positive cocycle (in the case $f^0 \geq 0$ and $f^1 = \bar{f}^2$, one gets $\omega(f^0, f^1, f^2) > 0$); of course we denote $\partial f = dz\partial_z f$ and $\bar{\partial} f = d\bar{z}\partial_{\bar{z}} f$. Let us define $df = (\partial + \bar{\partial})f$, then the functional $\widetilde{\omega}(f^0, f^1, f^2) = \int_\Sigma f^0 df^1 \wedge df^2$ is a cyclic cocycle (but is not positive).

2) Let M be an even dimensional oriented Riemannian manifold then

$$\tau(f^0, f^1, \ldots, f^n) = \int_M \operatorname{Tr}\left(\frac{1+\gamma_5}{2}\right) f^0 \partial\!\!\!/ f^1 \cdot \partial\!\!\!/ f^2 \cdot \ldots \cdot \partial\!\!\!/ f^n)\rho \tag{33}$$

(where ρ is the volume element, $\partial\!\!\!/ f = \gamma^\mu \partial_\mu f$ in the Clifford algebra and γ_5 is the helicity operator) is a positive cocycle whereas $\tilde{\tau} = \tau - \frac{1}{2} bB_0\tau$

$$\tilde{\tau}(f^0, f^1, \ldots, f^n) = \int_M f^0 df^1 \wedge df^2 \wedge \ldots \wedge df^n \tag{34}$$

is a cyclic cocycle.

Before ending this paragraph it is interesting and not too difficult to see how one can get a whole ascending hierarchy of cocycles from the data of a cyclic cocycle $\widetilde{\omega}$ of low dimensionality (for instance $\widetilde{\omega}_4 \in Z^4_\lambda$). We first choose φ_5 such that $B_0\varphi_5 = \widetilde{\omega}_4$ and such that $\omega_6 = b\varphi_5$ is a positive (even) cocycle, we then build the corresponding cyclic cocycle $\widetilde{\omega}_6$ and we iterate the construction. In this way we build a sequence of cyclic cocycles $(\widetilde{\omega}_4, \widetilde{\omega}_6, \widetilde{\omega}_8, \ldots)$ as well as a sequence $(\varphi_5, b\varphi_5, \varphi_7, b\varphi_7, \ldots)$ corresponding to the linear functionals

$$\varphi_p(a^0, \ldots, a^p) = T(a^0 q a^1 \ldots q a^p) \quad (p \text{ being odd}).$$

$$b\varphi_p(a^0, \ldots, a^{p+1}) = T(a^0 q a^1 \ldots q a^{p+1})$$

and T being a trace on QA. In order to build positive odd cocycles, one should use traces on the algebra $\epsilon(A)$.

7. ENTIRE CYCLIC COHOMOLOGY

We already mentioned in 4.3 that even cyclic cohomology could be reconstructed from the study of odd traces on QA *i.e.*, vanishing on the even part (and odd cyclic cohomology from odd traces on ϵA) but we did not really use this fact. Also, cyclic cohomology groups are graded by intergers: a given cyclic cocycle can be considered as an n-linear form but n is fixed and we did not construct any cocycle that would appear sometimes as a p-form and sometimes as a q-form. However, a trace on QA (or ϵA) is an object that can be considered as a form of any degree p when it is restricted to the domain spanned by $x_0 q(x_1) \ldots q(x_p)$ and $q(x_1) \ldots q(x_p)$. One would like to define a cohomology theory in such a way that cocycles appear as sequences $(\varphi_p)_{p \in N}$ where φ_p is a $(p+1)$-linear form on the algebra A. Such a cohomology would not be Z-graded but only Z_2-graded. All this motivates the definition of entire cyclic cohomology [25].

7.1. Functionals of arbitrary order and traces on QA and ϵA

At the purely algebraic level one can establish a canonical one to one correspondence between the following three notions on algebra A.

(i) cocyles with infinite support in the (b, B) bicomplex (discussed in Section 5.3) which are normalized.

(ii) linear functionals φ on the universal differential algebra ΩA such that

$$\varphi(\omega_1 \omega_2 - (-1)^{\partial_1 \partial_2} \omega_2 \omega_1) = \frac{1}{2}(-1)^{\partial_1} \varphi(d\omega_1 d\omega_2) \tag{35}$$

(iii) Odd traces on the Cuntz algebra QA or on Zekri algebra ϵA.

Several comments are in order: (i) the «infinite support» requirement means that we actually get functionals of arbitrary order. Calling C^n the space of continuous $n+1$-linear forms ϕ on A, we define C^{ev} and C^{odd} as follows $C^{ev} = \{(\phi_{2n})_{n \in N},\ \phi_{2n} \in C^{2n}\ \forall n \in \mathrm{N}\}$, $C^{odd} = \{(\phi_{2n+1})_{n \in N},\ \phi_{2n+1} \in C^{2n+1}\ \forall n \in \mathrm{N}\}$. The boundary operator $b + B$ maps C^{ev} to C^{odd} and C^{odd} to C^{ev}. The normalization condition is the following: a cocycle $(\phi_n)_{n \in N}$ is normalized iff, for any m the cochain $B_0 \phi_n$ is cyclic, *i.e.* iff $B_0 \phi_n = \frac{1}{n} A B_0 \phi_n$. The reason for imposing this condition is that only normalized cocycles have a natural interpretation in terms of ΩA, QA or ϵA. Fortunately, for every cocycle one can find a normalized cohomologous cocycle, so this is not a restriction. (ii) Remember that cyclic cocycles are graded traces (they vanish on elements of the form $\omega_1 \omega_2 - (-1)^{\partial_1 \partial_2} \omega_2 \omega_1)$, therefore the right-hand side of (iii) comes from the fact that φ is not «homogeneous» but has a support in all dimensions. In [20], [27], the theory is extended to Z_2-graded algebras, it is therein proposed to call para-brackets the kind of elements of ΩA on which φ vanishes (φ becomes a «paratrace»). (iii) The cohomology defined in (i) is not Z-graded but only Z_2-graded (this comes from the fact that $b+B$ is not homogeneous), the precise relation with QA and ϵA is the following: odd cocycles φ are given by odd traces T on QA and even cocycles φ are given by odd traces or ϵA. The relation between (ii) and (iii) should not be too surprising if we remember the relation (8) of Section 2.8.

7.2. Entire cyclic cohomology of A ♠

It can be proved that the cohomology defined by (i) of 7.1 is trivial! However, provided we control the growth of $||\phi_m||$ in a cochain (ϕ_{2p}) or (ϕ_{2p+1}) we get something non trivial in general and useful in order to analyse infinite-dimensional spaces (or cycles) such as those that one is confronted with in Quantum Field Theory. Here we suppose that A is a Banach algebra (to be able to define, for any m and $\varphi \in C^m$, the norm $||\varphi_m|| = \sup\{|\varphi_m(a^0,\ldots,a^m)|\,;\ ||a^j|| \le 1\}$. We are now ready for the following definition: a cochain of the (b,B)-mixed complex is called entire if the radius of convergence of the corresponding entire series is infinity. One gets a complex entire series $\Sigma||\phi_{2p}||\frac{z^p}{p!}$ for an even cochain (ϕ_{2p}), and a series $\Sigma||\phi_{2p+1}||\frac{z^p}{p!}$ in the case of an odd cochain (ϕ_{2p+1}). Entire cohomology is defined for entire cochains in 7.1. The corresponding cohomology groups are noted $H_\epsilon^{\mathrm{eve}}$ and $H_\epsilon^{\mathrm{odd}}$. The boundary operator on C^p is defined as $\partial = (p+1)b + \frac{1}{p}B$. One can check that if ϕ is an even (or odd) entire cochain, then so is $\partial\phi$ which makes the notion of entire cohomology meaningful. Notice that an entire cochain ϕ (not necessarily a cocycle) defines an entire function on the algebra A : $F_\phi(x) = \sum_{n=0}^{\infty}(-1)^N\phi_{2n}(x,x,\ldots,x)/n!$, $x \in A$. One can be tempted of interpreting the multilinear forms ϕ_{2n} as some kind of N-point functions in a Quantum Field Theory. Entire cocycles were also introduced in [29], [30] in a quite different context and have been particularly studied in the case of two-dimensional supersymmetric Wess Zumino models (one space, one time). An explicit expression for an entire cocycle is also given there (cf. also [28]). We will return to this in section 10.6.

We already mentioned in Section 5.4 that one can consider infinite-dimensional cycles $\Lambda = \otimes_{p=0}{}^{\infty}\Lambda^p$ over an algebra A (with a homomorphism $A^p \to \Lambda^0$). This is expected when A is «big» (like in Quantum Field Theory). Then a functional μ (generalizing $\int$) over the differential algebra (Λ, d) satisfying (ii) of 7.1 *i.e.*, $\mu(\lambda_1\lambda_2 - (-1)^{\partial_1\partial_2}\lambda_2\lambda_1) = \frac{1}{2}(-1)^{\partial_1}\mu(d\lambda_1 d\lambda_2)$, with $\lambda_i \in \Lambda$ should give us entire cocycles, via its character, as in 5.4. We will return to this problem in Section 10.6.

8. THE LODAY QUILLEN COHOMOLOGY OF THE LIE ALGEBRA OF MATRICES ♠

There is another approach to cyclic cohomology. This approach was followed by [30], independently from [7]. We have not followed this last point of view here but we will indicate what the relation between both constructions is.

Let A be an associative algebra and $M_n(A) = M_n \otimes A$ the algebra of matrices with coefficients in A ; it is also a Lie algebra for the commutator [ ].

One can build the universal differential algebra $\Omega(M_n(A))$. There is a map π : $\Omega(M_n(A)) \to \Omega(A)$ defined on monomials by

$$\pi(m_0 \otimes a_0 \delta(m_1 \otimes a_1)\delta(m_2 \otimes a_2) \ldots \delta(m_n \otimes a_n)) = \\ = \mathrm{Tr}(m_0 m_1 \ldots m_n) a_0 \delta a_1 \ldots \delta a_n. \tag{36}$$

If τ is a cyclic cochain on A , one can build a cyclic cochain Tr# τ on $M_n(A)$ as follows:

$$(\mathrm{Tr} \ \# \ \tau)(m_0 \otimes a_0, \ldots, m_n \otimes a_n) = \\ = \mathrm{Tr}(m_0 \ldots m_n)\tau(a_0, a_1, \ldots, a_n). \tag{37}$$

One can check, using the cyclicity of the trace Tr, that Tr # τ is indeed cyclic.

The above cyclic cochain Tr # τ is cyclicly antisymmetric (by construction) but is not fully antisymmetric. We want to build a fully antisymmetric form $\phi(\tau)$; we just need to antisymmetrize Tr# τ: $\phi(\tau) \doteq \mathrm{Ant}(\mathrm{Tr} \ \# \ \tau)$. Then $\phi(\tau)$ is an antisymmetric form on the Lie algebra $(M_n(A), [\,])$.

One knows to define in general the coboundary operator δ on antisymmetric forms on a Lie algebra: $\delta\psi$.

The theorem of Loday Quillen establishes a relation between cyclic cohomology $(b\tau = 0)$ and the Lie algebra cohomology of $M_n(A)$ $(\delta(\phi(\tau)) = 0)$. One finds in general

$$\phi(b\tau) = \delta\phi(\tau). \tag{38}$$

It was important, in the previous construction, of choosing τ , cyclic and of antisymmetrizing Tr # τ rather than τ alone since Tr($m_0 \ldots m_n$) is not antisymmetric. One could try to start with a Hochshild cochain σ , rather than with a cyclic cochain but in this case one would not get an equality between $\phi(b\sigma)$ and $\delta\phi(\sigma)$.

The previous relation involving b and δ does not lead actually to an isomorphism between cyclic cohomology and the full Lie algebra cohomology of $M_n(A)$ but only with the «primitive» part of it. This notion is somehow easier to define at the homological level. Let $\mathcal{L}$ be a Lie algebra, then one can consider the homology of the complex $(E_*(\mathcal{L}), d)$ where $E_n(\mathcal{L}) = \Lambda^n \mathcal{L}$ is the n-th exterior power of $\mathcal{L}$ and where d is defined as

$$d(x_1 \wedge \ldots \wedge x_n) = \\ = \sum_{1 \leq i \leq j \leq n} (-1)^{i+j} [x_i, x_j] n x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_n.$$

In the case where $\mathcal{L}$ is the algebra of infinite matrices which have only a finite number of nonzero entries in the algebra A. We write this complex as $(E_*(M(A)), d)$. We call then $E^{a\lambda}_{\mu\nu} = E_{\mu\nu} \otimes a_\lambda$ the matrix which has a_λ as the only nonzero entry on the (μ, ν) position; for $a_\lambda \neq 0$. The subspace of $(E_n(M(A), d)$ spanned by all elements $E^{a_1}_{i_1 i_2} \wedge E^{a_2}_{i_2 i_3} \wedge \ldots \wedge E^{a_n}_{i_n i_1}$ is noted $(PE_n(M(A))$. This defines actually a subcomplex $(PE_*(M(A)), d)$ called the subcomplex of primitive elements (the fact that it is indeed a subcomplex is the important observation). We call Prim($H_*(M(A))$) the homology of this subcomplex and one can prove that it is isomorphic with the cyclic homology of A.

To illustrate the above, we will consider the following example. Let τ be the 1-cyclic cocycle defined by $\tau(a, b) = \frac{1}{2}\,\mathrm{Tr}(a[F, b])$ where $a, b \in A = C(S^1)$ and F is the phase of the Dirac operator on the circle acting on $L^2(S^1)$ as follows $Dx(\theta) = -i\frac{\partial}{\partial\theta}\, x(\theta)$. Actually F is properly defined as $\lim_{\epsilon\to 0}$ phase $(D+\epsilon)_{\epsilon>0}$ since D has a zero mode (the constant function). Notice that F is diagonal (with eigenvalues +1 or −1) on the base $\{e^{in\theta}\}$. For a general manifold of dimension n, one would find $[F, b] \in \mathcal{L}^{n+\epsilon}$, $for \epsilon > 0$, but in the special case of the circle [35], one can take $\epsilon = 0$, so that $[F, b] \in \mathcal{L}^1$. The above cyclic cocycle is therefore well defined and the general theory (as well as an explicit calculation) shows that it is equal to $\frac{1}{2i\pi}\int_{S^1} a\,db$. If we replace the algebra A by the algebra $M_n(A)$ of $n\times n$ matrices over A and $\tau(.,.)$ by $\omega_n(.,.)$, with $\left[\omega_n(a\otimes(e_{ij})\,,\, b\otimes(e_{kl})) = \tau(a.b)\ \mathrm{Trace}\ ((e_{ij})(e_{kl}))\right]$ one can check (using $b\tau = 0$) that $\left[\omega([x, y], z) - \omega([x, z], y) + \omega([y, z], x) = 0\right]$ so that ω is a Lie algebra cocycle and defines a central extension of the Lie algebra $M_n(A)$ when we define a new brackett $[.,.]'$ as $[u, v]' = [u, v] + \omega(u, v)\,c$, the new generator, c, being in the center of the extension. This is, in a sense, the simplest kind of «Schwinger» term. Elements of $M_n(A)$ can be considered as loops in $M_n(\mathrm{C})$ so that the above cyclic cocycle is also responsible for the central extension of loop algebras and of loop groups [37].

9. THE NONCOMMUTATIVE ANALOGUES OF VECTOR BUNDLES

9.1. From vector bundles to projective modules of finite type

In usual (commutative) geometry, one introduces the notion of vector bundles. This notion is of fundamental importance in physics, namely in classical field theory since the «classical» matter fields are almost always described as sections of vector bundles. In the noncommutative framework, we have to generalize this notion. Actually, we have already given the clue: the important objects are not the vector bundles themselves but the space of their sections. From the algebraic point of vue, the space of sections of a vector bundle is a module over the (commutative) algebra of functions on the base of the vector bundle. By removing the adjective «commutative», we are led to the idea of replacing the vector bundles by modules over an algebra A. Actually, the space of sections of a vector bundle is not an arbitrary module: it is a projective module of finite type (these properties will be recalled below) and this will indeed be the right notion to generalize.

9.2. Modules over an algebra A (We assume that A is unital)

As it is well-known, a module over A is like a vector space with the difference that A is an algebra (actually a ring) and not a field. One can think of a complex module E as a complex vector space endowed with a representation of the algebra A. Actually, when A is not abelian, one has to distinguish between left and right A modules. For notational reasons, it is convenient to use right modules. As for vector spaces, one introduces the dual E^* of E as $E^* = \mathrm{Hom}_A(E, A)$; notice we can identify E with $\mathrm{Hom}_A(A, E)$ and A with $\mathrm{End}_A(A)$. This justifies the use of dyadic formalism (bra-ket notation) in this case: if $|\xi\rangle \in E$ and $\langle\varphi| \in E^*$ then $|\xi\rangle\ \langle\varphi| \in \mathrm{End}_A(E)$ and $\langle\varphi|\xi\rangle \in \mathrm{End}_A(A) = A$. The right action of A on E can be written $|\xi\rangle a = |\xi a\rangle$.

Notice that $A^n = \oplus_{i=1}^n A$ is a unital A-bimodule. Indeed $x\{a^1, \ldots, a^n\}y = \{xa^1y, \ldots, xa^ny\}$

E is called a free module if it is isomorphic to A^n. In this case, one can find a basis $|e_i\rangle$, i.e., a minimal generating family as well as a dual basis $\langle e^i|$ with the properties $\langle e^i \mid e_j\rangle = \delta^i_j$ and $|e^i\rangle\ \langle e_i| = 1$ (using Einstein's convention). A free module behaves as a vector space, n is called the dimension of E. Notice that the space of sections of a trivial vector bundle over M is a free module over $C(M)$.

E is a module of finite type whenever there exists a morphism (projection) $\pi : A^n \to E$. In this case, one can again find a base $\{|v_i\rangle\}$. This base is the image of the base $|e_i\rangle = \{0, 0, \ldots, 1, \ldots, 0\}$ in A^n under the morphism (projection) π. In other words, E is of finite type if it can be finitely generated (this, of course, does not imply that it is free!).

E is a projective module of finite type if 1) it is of finite type (hence we have a basis $|v_i\rangle = \pi|e_i\rangle$) and 2) it is projective, in the sense that there exists a lift $\lambda : E \to A^n$ such that $\pi\lambda = 1_E$. This last property allows us to build a dual basis $\langle v^i|$; we will have $|v^i\rangle\ \langle v_i| = 1$, as in the free case, but $\langle v^i \mid v_j\rangle \neq \delta^i_j$. Indeed, we build the dual basis $\langle v^i| = \tilde{\lambda}\langle e^i|$ where $\tilde{\lambda} : (A^n)^* \to E^*$ is the transpose of λ. The closure relation $|v^i\rangle\ \langle v_i| = 1$ is another way of writing $\pi\lambda = 1$ and the pseudo-orthogonality relation reads $\langle v^i \mid v_j\rangle = p^i_j \in A$ ($p^i_j \neq \delta^i_j$ in general), where p^i_j are the components of the projector $p = \lambda\pi$ of the free module A^n (p is indeed a projector since $pp = \lambda\pi\lambda\pi = \lambda 1 \pi = p$). Notice that $p = \lambda\pi \in \mathrm{End}_A A^n$ allows us to decompose A^n as follows

$$A^n = pA^n + (1-p)A^n$$

$$\lambda \uparrow \downarrow \pi$$

$$E$$

π (and λ) are isomorphisms between E and pA^n (it is clear that E is indeed a right module). Since $\mathrm{End}_A(A^n) = M_n(A) = M_n \otimes A$, we can represent p as a $n \times n$ matrix with elements in A. We could also caracterise such a finite projective module E by writing that it has to be a direct summand of a free module; this means that (up to isomorphism) we can write A^n as $A^n = E \oplus F$. It can be shown that, in this case F is also of finite type (but not necessarily free).

It can be shown that the space of sections of a vector bundle above a manifold M is always a projective module of finite type over the commutative algebra $A = C(M)$ of functions on M.

In the noncommutative framework, the notion of vector bundles (or better the notion of space of sections of a vector bundle) is replaced by the notion of projective modules of finite type over an algebra A, or equivalently, by projectors in the algebra $M_n(A)$.

9.3. K-theory of algebras A ♠

In differential geometry, one first defines the notion of vector bundles over a manifold X, and notices that the space of equivalence classes of vector bundles (under isomorphism) is an abelian monoid, one can define the sum of two vector bundles and this operation is associative. One is therefore tempted to construct a group by considering «negative» elements (exactly as when we construct the integers out of the positive integers). A technical complication is that, in the present case, the monoid is not simplifiable ($a+c = b+c$ does not imply $a = b$). There is nevertheless a simple way to get around this problem; following Grothendieck, one defines the abelian group $K^0(X)$ as the space of equivalence classes (a,b), where $(a,b) \sim (a',b')$ if and only if there exist c such that $a + b' + c = a' + b + c$, (morally $a - b = a' - b'$), a, b, a', b', c being themselves classes of isomorphism of vector bundles. Usual vector bundles can be written as $(a,0) = +a$ and «virtual» vector bundles as $(0,a) = -a$. The shortest way of defining the group $K^1(X)$ is to define it as the K^0 group of its suspension SX. The suspension of a space X is exactly what our intuition suggests (for instance the suspension of a circle is a two-sphere and more generally $SS^p = SS^{p+1}$). One could be tempted of continuing this way and defining $K^2(X) = K^1(SX) = K^0(SSX)$ but it turns out that $K^0(X) = K^0(S^2X)$ – Bott periodicity –: a space has the same K-theory as its double suspension [31] [32]. So, topological K-theory stops there and we have only to consider $K^0(X)$ and $K^1(X)$. Actually, we consider here only *complex* vector bundles, indeed the periodicity is not two but eight in the real case. The reader certainly knows that, even in the realm of «commutative» geometry, one can define higher «algebraic K-theory groups» or «Quillen groups»: their noncommutative counterpart also exists but we will not discuss them here.

In noncommutative geometry one follows the same construction and, because of the results of sect. 9.2, defines the K-theory groups $K_0(A)$, for a unital algebra A, as the abelian group associated to the isomorphism classes of finite projective modules over

A. Equivalently, it is also the abelian group associated to equivalence classes of idempotents e (projectors in $M_k(A)$). Again, the group $K_1(A)$ is defined as $K_0(SA)$ where the suspension of an algebra A is defined as follows (by «dualizing» the corresponding definitions for spaces): SA is the subalgebra of the algebra of continuous functions on [0,1], value of in A such that $f(0) = f(1) = 0$. There is actually a direct algebraic definition of $K_1(A)$ which is less intuitive: it is the quotent of $GL_\infty(A)$ by its commutator subgroup. The important observation making link with commutative geometry is that, in the case where $A = C(X)$, one gets $K_i(A) = K^i(X)$, $i = 0,1$ (Serre-Swann's theorem).

9.4. Connections on finite projective modules

As in (commutative) differential geometry, it is useful to introduce connections (and their curvatures). The basic ingredients are the following: A an associative algebra, E, a right finite projective module over A and $\Lambda = \oplus_{p=0}\Lambda^p$ a graded differential algebra, with $\Lambda^0 = A$ – or at least a homomorphism $A \to^\rho \Lambda^0$ as in 5.4, then Λ is a A-bimodule –. In the commutative case, A would be $C(X)$, E would be the space of sections of a vector bundle over X and Λ the algebra of differential forms. We will now define a notion generalizing the notion of «exterior differential acting on p-forms valued in a vector bundle» – Actually, in most cases, it is even more convenient to assume that Λ is a cycle over A, as defined in Sect. 5.4, which means that we have also a trace $\int : \Lambda^n \to \mathbb{C}$ such that $\int d\omega = 0$ for $\omega \in \Lambda^{p-1}$. One first considers the space $E_\Lambda = \oplus_{p=0}^n E_\Lambda^n$ with $E_\Lambda^p = E \otimes_A \Lambda_p$. Notice that elements of E_Λ^p should be considered as « p-forms valued in E »; notice also that E_Λ^p is still a right A-module. A Λ-connection ∇ on E – one may call it a covariant differential –, is a map from $E_\Lambda^o = E$ into E_Λ^1 such that

$$\nabla(Xf) = (\nabla X)f + X \otimes df \quad \text{where} \quad X \in E \quad \text{and} \quad f \in A. \tag{40}$$

Notice that ∇ is $\mathbb{C}$-linear but not A-linear. The covariant exterior differential on E is the graded derivation of E_Λ which extends ∇, *i.e.*, one imposes the (graded) Leibnitz rule

$$\nabla(X \otimes \lambda) = (\nabla X)\lambda + (-1)^p X \otimes d\lambda \quad \text{with} \quad X \in E_\Lambda^p, \lambda \in \Lambda. \tag{41}$$

In order to introduce covariant derivatives «in a given direction,» one needs a kind of dual L of Λ^1 (over the algebra A), indeed, if $x \in E$ and $\xi \in L$ then $\nabla X \in E \otimes_A \Lambda^1$ and we define

$$\nabla_\xi X = \langle \nabla X, \xi \rangle \in E. \tag{42}$$

A particularly important case occurs when one starts with E , A and a Lie algebra L acting by derivations on A , so that $L \subset DerA$. In the case where A has enough derivations, it can be usefull to consider the space $C(DerA, A)$ of cochains of $DerA$ valued in A. This space has naturally a structure of graded differential algebra and one can define connections by chosing Λ equal to $C(DerA, A)$. This space is however rather big (even in the classical case where A is $C(M)$, it contains highly non local objects) and it can be interesting to consider the smallest subalgebra of it that contains A; let us call it $\Omega_D(A)$. This new algebra is another possible generalization of usual differential forms; one can define connections by chosing Λ equal to $\Omega_D(A)$. This method is described in [47]. Notice that in this last case, one can introduce covariant derivatives in the direction of a derivation of A. One can also distinguish the following cases:

(i) $A = C(X)$ and L is the Lie algebra of vector fields on X. This is the case of differential geometry (and «linear connections» deal with the case where E itself is some tensorial power of the tangent bundle TX or of the cotangent bundle T^*X.

(ii) (A, G, α) is a dynamical system, *i.e.* G is a Lie group acting in the algebra A by endomorphisms ($k \in G, f \in A, \alpha_k(f) \in A$). Then $LieG$ acts also on A by derivations: to each $\xi \in$ Lie G, one associates a derivation of A noted d_ξ. One then choose again $L =$ Lie G and build Λ as the exterior algebra over L^*. One obtains in particular $\nabla_\xi(xf) = (\nabla_\xi)f + X d_\xi f$. The study of connections on dynamical systems can be found in [34]. The case where A is a graded commutative algebra and L a graded Lie algebra of graded derivation is investigated in [17]. Let us return to the general case and define the curvature operator of the connection ∇ as ∇^2. ∇ is not linear, but we can easily check that, exactly as in the case of vector bundles, ∇^2 is a endomorphism of E_Λ; *i.e.*, ∇^2 is a linear operator: $\nabla^2(X\lambda) = \nabla^2(X)\lambda, x \in E_\Lambda, \lambda \in \Lambda$.

Since $(\Omega(A), \delta)$ is a universal object, it is enough to consider generalized differential forms (elements of $\Omega(A)$) valued in a right A-module E. E is, in the present cases, isomorphic to pA^n, this suggests that we should study the case $E = A$ and $\Lambda = \Omega(A)$. Here we suppose that A is unital, call 1 its unit and assume that $\delta 1 = 0$ in $\Omega(A)$. This means that A should be denoted $\widetilde{A}$ to agree with the notations of §2. Let ∇ be a Ω-connection on A. Take $1 \in A$, then we call $\omega = \nabla 1 \in \Omega^1(A)$, the connection one form. Take $f \in A$, then, writing $\nabla f = \nabla(f1) = (\nabla f)1 + f \otimes \delta 1 = \nabla f$ does not bring anything new. However, we can write $\nabla f = \nabla(1f) = (\nabla 1)\delta + 1 \otimes \delta f$, which shows that

(43) $$\nabla f = \delta f + \omega f \in \Omega^1$$

The curvature Θ is defined as

(44) $$\Theta = \nabla\omega = \nabla^2 1$$

Therefore $\Theta = \nabla(1\omega) = (\nabla 1)\omega + 1\delta\omega$ so that

(45) $$\Theta = \delta\omega + \omega^2$$

In the case of the commutative algebra $A = C^\infty(X)$, an Ω-connection is defined by an element

$$\omega = \sum_i a_i \delta b_i$$

of Ω^1, so that it can be represented as a function of two variables

$$\omega(x,y) = \sum_i a_i(x)(b_i(y) - b_i(x)).$$

Let us call « Cl », the «classical» universal map from $\Omega(A)$ to $\Lambda(X)$. Then

$$Cl(\omega) = \sum_i a_i db_i$$

$$Cl(\omega)(x) = \sum_i a_i(x)\partial_\mu b_i(x) dx^\mu.$$

In the case where X is a Riemannian manifold, let us call π the universal map factorizing the derivation $\not{d}$ from A to the Clifford algebra of the tangent bundle. Then

$$\pi(\omega) = \sum_i a_i \not{d} b_i.$$

Notice that there are many more Ω-connections than «classical» connections. For instance the element

$$\omega = f\delta f - \frac{1}{2}\delta(f^2) = \frac{\partial\omega(x,y)}{\partial y^\nu}/_{y=x} dx^\nu.$$

is not zero in $\Omega(A)$ but $Cl(\omega)$ and $\pi(\omega)$ are both zero. Moreover $\delta\omega = \delta f \delta f \in \Omega^2(A)$ is not zero either, but although $Cl(\delta\omega) = 0$ in $\Lambda(X)$, we see that $\pi(\delta\omega) = \not{d} f \; \not{d} f = \| df \|^2 \neq 0$ in Cliff (TX).

Let us compute $\Theta, \pi(\Theta)$ and $Cl(\Theta)$ in this case. In order to use standard notations, we call $A = Cl(\omega)$ and $\not{A} = \gamma^\mu A_\mu = \pi(\omega)$. Notice first that $\pi(\omega^2) = \not{A}\not{A} = A_\mu A^\mu = A^2$. Then

$$\pi(\delta\omega) = \sum_i \gamma^\mu \partial_\mu a_i \gamma^\nu \partial_\nu b_i = \sum_i \gamma^\mu\gamma^\nu \{\partial_\mu(a_i\partial_\nu b_i - a_i\partial_\mu\partial_\nu b_i)\}$$

It is convenient to introduce the symmetric tensor

$$B_{\mu\nu} = \sum_i a_i \partial_\mu \partial_\nu b_i$$

Then

$$\pi(d\omega) = \gamma^\mu \gamma^\nu \partial_\mu (A_\nu) - \gamma^\mu \gamma^\nu B_{\mu\nu}$$

On the other hand, calling

$$F = Cl(\Theta) = Cl(\delta\omega) = dA, \quad \text{and} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

the classical curvature of A, we may write

$$\gamma^\mu \gamma^\nu (\partial_\mu A_\nu) = \frac{1}{2} F_{\mu\nu} \gamma^\mu \gamma^\nu + \partial^\mu A_\mu$$

so that

$$\pi(\Theta) = \frac{1}{2} F_{\mu\nu} \gamma^\mu \gamma^\nu + \partial^\mu A_\mu - \gamma^\mu \gamma^\nu B_{\mu\nu} + A^\mu A_\mu$$

But $B_{\mu\nu}$ is symmetric so that we may call b the scalar

$$b = \gamma^\mu \gamma^\nu B_{\mu\nu} = g^{\mu\nu} B_{\mu\nu}$$

The final result is

$$\pi(\Theta) = \frac{1}{2} F_{\mu\nu} \gamma^\mu \gamma^\nu + (\partial^\mu A_\mu + A^\mu A_\mu - b). \tag{46}$$

This shows clearly that there is «more» in the connection a than in its classical counterpart A. At this point, the reader could be tempted to consider the expression $\mathrm{Tr}\ \pi(\Theta^2)$ in the Clifford algebra. Notice that Θ^2 itself is an element of $\Omega^4(A)$. A straightforward calculation leads to

$$\frac{1}{8} \mathrm{Tr}\ \pi(\Theta^2) = -\frac{1}{4} F_{\mu\nu} F^{\nu\nu} + \frac{1}{2} (\partial A + A^2 - b)^2. \tag{47}$$

The physicist recognises the Maxwell Lagrangian for electrodynamic (the field b does not propogate: it can be eliminated by using its equation of motion $\frac{\delta \mathcal{L}}{\delta b} = 0$ so that $b = \partial \cdot A - A^2$).

From the conceptual point of vew, what we just did was to use the universal map $\Omega(A) \to$ Cliff (TX) factorizing the derivation $d\!\!/ : A \to$ Cliff (TX) and to use the trace in the Clifford algebra. Notice that we used the trace and not the «supertrace» Str $(x) =$ tr $(\gamma_5 x)$; this would have lead to the topological invariant $F_{\mu\nu} \tilde{F}^{\mu\nu}$ rather than to the Maxwell lagrangian.

We will indicate in Sect. 12 how to couple this «electromagnetic» connection to «spinors» in the framework of noncommutative geometry.

We will not continue further the study of these Λ-connections; the interested reader is referred to [7]. A relation between these non classical connections and the geometric structure of canonical commutation relations has been studied in [45],[46]. Before ending this paragraph, we should mention that if $\Lambda = \oplus_{p=0}^{n}$ is a cycle over A, with $n = 2m$, even, then $\int \Theta^m$ is independent of the choice of the connection ∇; therefore, as in the case of differential geometry, one is tempted to consider

$$\frac{1}{m!}\int\left(\frac{\Theta}{2i\pi}\right)^m$$

as a characteristic number of the algebra A, obtained from the pairing of a finite projective module E and a cycle Λ. This will be done in the following. We will return to some aspects of connections in Sect. 12.

9.5. The pairing between even cyclic cohomology and $K_0(A)$

We know, from sect. 9.3 that a given finite projective module E determines a well defined element $[e]$ in $K_0(A)$; we also know that $[e]$ can be represented as a projector e in the space $M_k(A)$ of $k \times k$ matrices with elements in A. Besides, we know that if $[\varphi] \in H_\lambda^{2m}(A)$ is represented by a cyclic cocyle φ of order $2m$ on A, we may replace both A and φ by $M_k(A)$ and $\varphi\#\ \mathrm{Tr}$ (Morita invariance). The last remark of 9.4 suggests that we should consider the pairing

$$\langle [e],[\varphi]\rangle = \frac{1}{(2i\pi)^m}\frac{1}{m!}(\varphi\#\ \mathrm{Tr})(e,e,\ldots,e) \tag{48}$$

One can prove [7] that this indeed defines a pairing between $K_0(A)$ and $H^{\mathrm{eve}}(A)$ *i.e.*, the number on the left hand side does not depend on the choice of e in $[e]$ and of φ in $[\varphi]$. Moreover, one proves that

$$\langle [e],[\varphi]\rangle = \langle [e],[S\varphi]\rangle \tag{49}$$

where S is the periodicity operator (cf. sect. 5), and that, if φ is a cyclic cocycle defined as the character of a cycle Λ over A (sect. 5.4) then, one obtains

$$\langle [e],[\varphi]\rangle = \frac{1}{m!}\int\left(\frac{\Theta}{2i\pi}\right)^m. \tag{50}$$

Where Θ is the curvature of any Λ-connection on the module associated with e, as described in sect. 9.4. Notice that (eq. 49) shows that $K_0(A)$ actually pairs with periodic cyclic cohomology as defined in sect. 5.3. The reader should not be surprised that only the even part of $H_\lambda^*(A)$ pairs with $K_0(A)$ – this can be seen in the previous formula since Θ is an even dimensional object – The odd part plays an important role: it pairs with $K_1(A)$.

9.6. The pairing between odd cyclic cohomology and $K_1(A)$ ♠

We mentioned in 9.3 that $K_1(A)$ could be defined as the quotient of $GL_\infty(A)$ by its commutator subgroup. We will give the formula establishing the above mentioned pairing [7]. Let $u \in GL_k(A)$ be a representative of $[u] \in K_1(A)$ and $\varphi \in Z_\lambda^{2m-1}(A)$ a representative of $[\varphi]$. Then one defines

$$
\begin{aligned}
\langle [u],[\varphi]\rangle = {} & \frac{1}{(2i\pi)^m}\frac{1}{2^{2n+1}}\frac{1}{(m-\frac{1}{2})\dots\frac{1}{2}}\cdot \\
& \cdot(\varphi\#\,\mathrm{Tr})(u^{-1}-1,u-1,u^{-1}-1,\dots,u-1).
\end{aligned}
\tag{51}
$$

One can prove that this is indeed independent of the choice of the representatives within their equivalence class and that

$$\langle [u],[\varphi]\rangle = \langle [u],[S\varphi]\rangle. \tag{52}$$

Notice that u can be considered as a finite projective module for the suspension of the algebra A (this amounts to add another «dimension» to the problem – that physicists could be tempted of calling «time» –).

9.7. The pairing between $K_0(A)$ and entire even cyclic cohomology ♠

In the same way, one expects a pairing between $K_0(A)$ and $H_\epsilon^{\mathrm{eve}}$. Indeed, if φ is an entire cocycle, we can associate with it an entire function F_φ on A (cf. sect. 7.2). If e is a projector in A characterising a finite projective module, we may consider the pairing

$$\langle [e],[\varphi]\rangle = F_\phi(e) \tag{53}$$

between $K_0(A)$ and $H_\epsilon^{\mathrm{eve}}(A)$; as expected, one can indeed prove that the result is independent of the choice of e and φ within their respective equivalence classes. Moreover, we know (from sect. 7.1) that there is an equivalence between entire cocycles φ and odd traces τ on the Zekri algebra ϵA; it can be shown [25] that, it terms of τ, the previous result reads

$$\langle [e],[\varphi]\rangle = \tau\left(\frac{Fe}{\sqrt{1-(qe)^2}}\right) \tag{54}$$

where F now denotes the odd generator of ϵA over QA ($F^2 = 1$, of sect. 2.8) and q is the pseudo differential introduced in sect. 2.8, eq. 8.

10. FREDHOLM MODULES

10.1. Motivations

From the physical point of view, Classical Field Theory usually involves the data of a differential (or pseudo differential) operator P, for instance the Dirac operator, mapping the sections of a first vector bundle E_0 above X (the classical spinor fields, for example) into the sections of another bundle E_1 above X. Of course E_0 may coincide with E_1. Actually, physicists usually assume that the vector bundles E_0 and E_1 are equipped with some chosen metric (in order to write an action principle and expressions like $\int_M \langle \varphi, P\varphi \rangle)$. The first thing to do is then to complete the space of sections of E_0, E_1 for some Sobolev norm, thus building two Hilbert spaces H_0 and H_1 with $P : H_0 \to H_1$. One then wants to use a propagator, therefore an inverse Q to P; there may be problems at this level.

It is convenient to write $H = H_0 \oplus H_1$ and

$$F = \begin{bmatrix} 0 & Q \\ P & 0 \end{bmatrix}$$

and to think of (H, F) as a whole. The geometrical (or algebraical) structure given by a pair (H, F) is called a Fredholm module; we will give below (in Sect. 10.2) a precise definition. In the above case $A = C(X)$ and this commutative algebra is represented by multiplication operator in H.

Finally, when a physicist quantizes a classical theory and builds a quantum field theory, he follows well defined algorithmic procedures (that we will not recall here) that have certainly a purely geometrical interpretation since they depend only on geometrical data but these procedures have usually been developed in a perturbative context and their global geometrical meaning is often unclear. The notion of Fredholm module exists also when A is not commutative. The hope is that some of the material presented here may help to clarify these quantum aspects. It is believed – at least some people including the author believe – that such structures (or more probably generalizations of them) will give us one day a tool to analyse (and define) Quantum Field Theories in a nonperturbative way.

From the mathamatical point of view, and as we shall see below, Fredholm modules are also of fundamental importance. We already met, in the previous chapters, the noncommutative analogue of differential forms and of de Rham cohomology but not the analogue of elliptic operators. Elliptic operators are in a sense, dual to vector bundles: out of an elliptic operator and a vector bundle, one gets an integer, namely the index of this operator. In this sense, one can say that elliptic operators on a space X should allow us to build the «dual» of the theory of vector bundles above X. The latter being called K-theory (cf. Sect. 9.3), the former is then called K-homology. In order to define this notion properly in the noncommutative case, one defines Fredholm modules as in 10.2. (they are the proper generalization of the notion of elliptic operator). In the case where the algebra A under study is $C(X)$, the K-homology of X is roughly speaking defined as the space of homotopy classes of Fredholm modules over $C(X)$ –. As we shall see later (sect. 10.3), a Fredholm module called (H, F) allows us to build infinitely many cycles on A (in the sense of sect. 5.9); all have the same underlying differential algebra Λ but they are of dimension $n \geq p$ where p is a real number (not necessarily an integer) depending upon (H, F). The characters of those cycles give us, as in 5.4, a whole hierarchy of cyclic cocyles on the algebra A.

Fredholm modules being the generalization of elliptic operators, we will define their index in sect. (10.4). Since Fredholm modules yield cyclic cocyles and since cyclic cocycles pair with K-theory (cf. sect. 9.5), then Fredholm modules also pair with K-theory but, this time the value of the pairing will be an integer (since it will be equal to the index of an operator); this will be discussed in sect. 10.5. Till now, we did not mention the fact that, as it occurs most of the time in physics, the operator that we want to study is not necessarily bounded and does not have always an inverse; we will indicate in 10.6 how to handle this situation (as physicists know, one just has to introduce an extra-dimension). A last problem that appears in physical situations is that very often the Fredholm modules of interest are not finitely summable (the number p mentioned above is infinite); to tackle this situation, one introduces the notion of Θ-summable Fredholm modules and replace cyclic cohomology by entire cyclic cohomology; this will be discussed in 10.6.

10.2. p-summable Fredholm modules

We first recall a few basic definitions.

Polar decomposition. Let H a separate Hilbert space and $\mathcal{L}(H)$ the space of bounded operators on H. The first thing to remember is that if $T \in \mathcal{L}(H)$ then one can define the adjoint T^* of T and T^*T is a positive operator ($\forall x \in H, (T^*Tx, x) \geq 0$). One then defines $|T| = \sqrt{T^*T}$ – one can take the square root of a positive operator – indeed there is only one operator $|T|$ such that $|T|^2 = T^*T$. One can then write $T = |T|\phi_T$ where ϕ_T is by definition the phase of T.

Schatten classes $\mathcal{L}^p$. Let p be a real number $p \geq 1$, then, the Schatten class $\mathcal{L}^p$ is defined as the space of all bounded operators on H such that Trace $|T|^p$ is finite. Calling $\mu_n(T)$ the nth eigenvalue of $|T|$, one can replace the above by the condition that

$$\sum_{n=0}^{\infty} (\mu_n(T))^p$$

is finite. Intuitively, the eigenvalues of T should decrease fast enough at infinity and, using the quantum field theoretical jargon, one could say that p measures the way T behaves by power counting. One can prove that $\mathcal{L}^p$ is a two-sided ideal in $\mathcal{L}(H)$, that $\mathcal{L}^p \subset \mathcal{L}^q$ if $p \leq q$ and that $\mathcal{L}^p$ is complete for the Schatten norm $\| T \|_p = (\text{Trace } |T|^p)^{1/p}$. We should remember that

$$\mathcal{L}^1 \subset \mathcal{L}^2 \subset \ldots \subset \mathcal{L}^p \subset \ldots \subset \mathcal{L}^\infty$$

where $\mathcal{L}^1$ are trace-class operators, $\mathcal{L}^2$ are Hilbert-Schmidt operators, and $\mathcal{L}^\infty$ are compact operators; we refer to any treaty of functional analysis for a more detailed study of these spaces. Let us formally remember the definition of a Fredholm operator: $P \in \mathcal{L}(H)$ is Fredholm iff it is invertible modulo compact operators, *i.e.*, if one can find $Q \in \mathcal{L}(H)$ such that $PQ - 1$ and $QP - 1$ are compact operators.

Pre-Fredholm modules (H, F) over an algebra A (a possibly Z_2-graded-algebra). This is a Z_2-graded Hilbert space $H = H_+ \oplus H_-$ endowed, from the one hand, with a representation ρ of A into H, (so H is a left A-module) and, from the other hand, with a bounded operator $F \in \mathcal{L}(H)$ which is odd for the Z_2 grading and such that for any $f \in A$, the operators $\rho(f)(F^2 - 1)$ and $[F, \rho(f)]$ are compact. The grading operator can be written as

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Any operator G of $\mathcal{L}(H)$ can be written as $G = G_+ + G_-$, where

$$G_\pm = \frac{1}{2}\left(G \pm G^\Gamma\right)$$

with $G^\Gamma = \Gamma G \Gamma$. The requirement for F to be odd means that $F^\Gamma = -F$, thus F can be written as

$$F = \begin{bmatrix} 0 & Q \\ P & 0 \end{bmatrix}.$$

As an example at the classical level, one could think of P as an elliptic operator of order 0 from a vector bundle E_+ to a vector bundle E above the same space X; H_+ and H_- are then the Hilbert spaces of square integrable sections of these bundles and $f \in A = C(X)$ acts on H_+ and H_-, by multiplication. A similar kind of structure naturally emerges in Quantum Field Theory as well (the case of supersymmetric Wess-Zumino models, where F is a Dirac operator on a loop space – a supersymmetry charge – and F^2 is the Hamiltonian investigated in [29]).

Fredholm modules (H, F) over an algebra A. The definition is the same as for a pre-Fredholm module but we replace the condition « $\rho(f)\ (F^2 - 1)$ is compact» by the condition « $F^2 = 1$ ». In order to use the formalism of sections 1 to 9, it is indeed important to have $F^2 = 1$ (and not only $F^2 - 1$ compact!), this is linked, as we shall see later, with the fact that we want to define an operator d with $d^2 = 0$. Actually there is a way to associate canonically to each pre-Fredholm module a Fredholm module, this is explained in [7], p. 305, let us just indicate that one has just to double the number of components, *i.e.*, to add an extra-dimension (time?) *i.e.*, to replace $H = H_+ \oplus H_-$ by

$$\tilde{H} = H \hat{\otimes} (\mathbf{C} \oplus \mathbf{C}) = (H^+ \oplus H^-) \oplus (H^- \oplus H^+).$$

The first notion of Fredholm module is due to Atiyah [36], in the case of manifolds of even dimensions, his definition has been generalized by several authors and finally brought to the arena of noncommutative geometry in [7].

p-summable pre-Fredholm modules (H, F) over an algebra A. In the case of a pre-Fredholm module one imposes both $[F, \rho(f)] \in \mathcal{L}^p$ and $\rho(f)[F^2 - 1] \in \mathcal{L}^p$. In the case of a Fredholm module one imposes only the first condition since the second one is already replaced by the condition $F^2 = 1$. Example: The Fredholm module coming from a given elliptic operator acting on sections of a vector bundle over X is p-summable for $p > \dim X$.

We will see later that it may be necessary to remove the condition that F is a bounded operator and replace it by the weaker hypothesis that F is a possibly unbounded, self-adjoint operator such that $[F, \rho(f)]$ is bounded for any $f \in A$ (morally F is of degree 1) and such that F^{-1} is p-summable (morally, the eigenvalues of F increase rapidly enough at infinity). Such a data (H, F) can be called an «unbounded Fredholm module» or a «Connes module» (as in [8]). Actually one can even get rid of the hypothesis that F is invertible. The typical example is given by the Dirac operator acting on L^2 sections of the bundle of spinors over a Riemannian spin manifold.

10.3. From p-summable Fredholm modules to cyclic cohomology

The fundamental observation allowing us to link Fredholm modules with the rest of the theory developed in sections 1 to 9 is the following: let (H, F) a Fredholm module with grading Γ, let $x \in \mathcal{L}(H)$, then define

(55) $$dx = i[F, x]_g$$

where the graded commutation is defined as follows

(56) $$[F, x]_g = \Gamma[\Gamma F, x] = (Fx - \Gamma x \Gamma F) = (Fx - x^\Gamma F).$$

Then d is a derivation and

$$d^2 = 0$$

Actually, one obtains more generally the obvious equivalence

(57) $$d^2 = 0 \Leftrightarrow F^2 = 1$$

Notice that one can always write, for $x \in \mathcal{L}(H)$

$$x = x_+ + x_-$$

with

$$x_\pm = \frac{1}{2}(x \pm x^\Gamma)$$

and $x^\Gamma = \Gamma x \Gamma$. Then dx defined as above can also be written

$$dx = \underbrace{[iF, x_+]}_{dx-} + \underbrace{\{iF, x_-\}}_{dx+}.$$

When $F^2 \neq 1$ we can check that $d^2 x = [x, F^2]$.

One can therefore apply the universal properties (cf. sect. 2) of the differential envelope $\Omega(A)$ and associate the «abstract» monomials $a_0 \delta a_1 \delta a_2 \ldots \delta a_n$ of $\Omega(A)$ with the «concrete» operators

$$a_0 da_1 da_2 \ldots da_n = i^n \rho(a_0)[F, \rho(a_1)]_g [F, \rho(a_2)]_g \ldots [F, \rho(a_n)]_g.$$

All the constructions carried out in sections 1 to 9 could be «explicitly» done by representing $\Omega(A)$ is such a way.

At a formal level, the idea is the following. One first builds the direct sum

$$\Lambda = \bigoplus_{q=0}^{\infty} \Lambda^q$$

where $\Lambda^0 = \rho(A)$ and Λ^q is the linear span in $\mathcal{L}(H)$ of monomials $\rho(a_0) d\rho(a_1) \ldots d\rho(a_q)$ and where $d\rho(a_i) = i[F, \rho(a_i)]_g$. Then Λ is a differential algebra with differential d. Moreover, one can define the following «supertrace»

$$\text{(58)} \qquad \mathrm{Str}(x) = \mathrm{Tr}\ \Gamma x$$

This definition makes sense if $x \in \mathcal{L}^1$ (a trace-class operator). Notice that $\mathrm{Str}(xy) = (-1)^{pq}\,\mathrm{Str}\,(yx)$ where $p = \deg x$, $q = \deg y$ and that

$$\mathrm{Str}(dx) = \mathrm{Tr}\ \Gamma dx = i\,\mathrm{Tr}(\Gamma F x - \Gamma x \Gamma F) =$$

$$= i\,\mathrm{Tr}(\Gamma F x - x \Gamma F) = 0.$$

So that Str is a closed graded trace. Actually, one proves that if (H, F) is p-summable, then $\Lambda^k \subset \mathcal{L}^{p/k}$. We are therefore in the situation described in sect. 5.4. We obtain a cycle $\omega \in \Lambda^n \to \mathrm{Str}(\omega)$ when n is big enough such that the supertrace converges, and the character of this cycle yields a cyclic cocycle. Notice however that the supertrace vanishes if x is homogeneous with odd degrees, therefore we will only get even cyclic cocycles in this way (this could be called the Furry's theorem of noncommutative geometry!). Since F is odd (remember the definition of Fredholm modules), we find that, formally, we can write

$$\mathrm{Str}(x) = \frac{1}{2}\,\mathrm{Tr}(\Gamma F[F,x]) \tag{59}$$

but this makes sense for any bounded opeartor x as soon as $[F,x]$ is trace class. Using the fact that the Fredholm module (H,F) is p-summable, one finds that

$$\mathrm{Str}(\rho(a_0)\,d\rho(a_1)\ldots d\rho(a_n))$$

exists whenever $n \geq p-1$. In other words, to each p-summable Fredholm module (H,F), one can associate a hierarchy of even cyclic cocycles τ_n, $n = 2m$, obtained as the characters of the cycles

$$\int \omega = (2i\pi)^m m!\ \mathrm{Str}(\omega), \tag{60}$$

for $\omega \in \Lambda^n$, (cf. sect. 5.4), and explicitly given by any of the following formulae:

$$\begin{aligned}\tau(a_0,a_1,\ldots,a_n) &= (2i\pi)^m m!\ \mathrm{Str}(\rho(a_0)\,d\rho(a_1)\ldots d\rho(a_n))\\ &= (2i\pi)^m m!\ \mathrm{Tr}\left(\Gamma\rho(a_0)\,d\rho(a_1)\ldots d\rho(a_n)\right)\\ &= (2i\pi)^m m!\ i^n\ \mathrm{Tr}\left(\Gamma\rho(a_0)\,[F,\rho(a_1)]_g\ldots[F,\rho(a_n)]_g\right)\\ &= (2i\pi)^m m!\frac{i^{n-1}}{2}\ \mathrm{Tr}\left(\Gamma F\,[F,\rho(a_0)]_g\ldots[F,\rho(a_n)]_g\right).\end{aligned} \tag{62}$$

The last formula results from the formal trick (58,59) and this actually shows that one can indeed take $n \geq p-1$ (the fact that the trace converges for $n \geq p$ resulting obviously from the definition of p-summability). These cyclic cocycles τ_n are called the «characters of Fredholm module (H,F) and their cohomology classes $[\tau_n]$ are denoted $Ch^m(H,F)$. Notice that if A is trivially Z_2-graded, then $\rho(a), a \in A$ is even $(\rho(a)^\Gamma = \rho(a))$ and the graded commutator becomes a usual commutator: $d\rho(a) = i[F,\rho(a)]$.

The example where $A = C^\infty(X)$, X a two-torus (considered as a quotient of the complex plane by a lattice) and

$$F = \begin{bmatrix} 0 & (\partial+\epsilon)^{-1} \\ \bar\partial+\epsilon & 0 \end{bmatrix}$$

is carried out in [7]. Here ϵ is chosen to make $(\bar{\partial} + \epsilon)$ invertible. Since the space is two-dimensional, the module is p-summable as soon as $p > 2$; it is in particular 3-summable and one gets a two-cocycle

$$\tau(f^0, f^1, f^2) = (2i\pi) \operatorname{Str}\left(f^0 i\,[F, f^1] i [F, f^2]\right).$$

The explicit calculation shows that

$$\tau(f_0, f_1, f_2) = \int f^0\, df^1 \wedge df^2,$$

where $f^i \in C^\infty(X)$. The result is not too surprising in view of the correspondence between cyclic cohomology and the De Rham homology for currents mentioned in sect. 4.6.2.

Since $F^2 = 1$, notice that

$$[F, a^0] F = F a^0 F - a^0 = F a^0 F - F^2 a_0 = F[a^0, F]$$

(Here our notation does not distinguish between a and $\rho(a)$). Since $F = F^{-1}$ we could write (for example)

$$\begin{aligned}
&\Gamma F^{-1}[F, a^0] F^{-1}[F, a^1] F^{-1}[F, a^2] = \\
&= \Gamma F[F, a^0] F][F, a^1] F[F, a^2] = \\
&= \Gamma F[F, a^0] F^2 [F, a^1][F, a^2] = \\
&= \Gamma F[F, a^0][F, a^1][F, a^2].
\end{aligned}$$

In the case where A is trivially Z_2-graded , one can therefore also write the previous cyclic cocyles as follows:

$$\begin{aligned}
&\tau(a^0, a^1, \ldots, a^n) = \\
&= (2i\pi)^m m! \frac{1}{2} \operatorname{Tr}\left(\Gamma F^{-1}[F, a^0] F^{-1}[F, a^1] \ldots F^{-1}[F, a^n]\right).
\end{aligned} \tag{63}$$

This may appear as an artificial and rather formal manipulation but it is useful for the following reason: one can prove that for an invertible operator F of square not equal to 1, this formula still gives a cyclic cocycle. One does not even have to suppose that F is a bounded operator on the Z_2-graded Hilbert space H but only that F^{-1} is bounded, that $F^{-1}[F, \rho(a)]$ belongs to $\mathcal{L}^p$ for any $a \in A$ and, of course that $F\Gamma = -\Gamma F$. Such a data is called a «Connes module» in [8]. The proof of this result [7] is obtained by building, out of this Connes module, two Fredholm modules (H_1, F_1), (H_2, F_2), with corresponding characters τ_1, τ_2 and getting τ has

$$\frac{1}{2}(\tau_1 - \tau_2).$$

One can even replace the condition « F-invertible » by the weaker condition

$$(1 + F^2)^{-1} \in \mathcal{L}^{p/2}.$$

In the last case, the previous formula is still valid, provided we regularise F^{-1}; this is obtained by replacing F by $F_\epsilon = \epsilon F \hat{\otimes} 1 + 1 \hat{\otimes} \beta$. where

$$\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and ϵ is a small real parameter. Indeed $F_\epsilon^2 = (1 + \epsilon^2 F^2) \hat{\otimes} 1$ so that F_ϵ is invertible. One can think of ϵ as the inverse of a Pauli-Villars mass regulator, indeed, we could also replace F by $F_M = F \hat{\otimes} 1 + M 1 \hat{\otimes} \beta$ then $F_M^2 = (F^2 + M^2) \otimes 1$ and this is – in a particular case – the well-known construction associating to the Dirac operator, the Dirac Hamiltonian with mass M. In any case, formula (63) still gives us a cyclic cocyle τ_ϵ when we replace F and F^{-1} by F_ϵ and F_ϵ^{-1}. The important fact [7] is that the cohomology class of τ_ϵ is independent of ϵ and that the limit

$$\tau(a^0, a^1, \ldots, a^n) = \lim_{\epsilon \to 0} \tau_\epsilon(a^0, a^1, \ldots, a^n)$$

exists. The limit $\epsilon \to 0$ (or $M \to \infty$) can be thought of as a case where everything is localized. The reader familiar with one loop calculations in Quantum Field Theory can recognise the previous cocycle as a one loop Feynman diagram of a special type. In the case where F is the Dirac operator on an even dimensional manifold, τ appears as a fermionic loop (a $(n + 1)$-gone) with one insertion of the helicity operator ($\Gamma = \gamma_5$ here). Formally, if A is the algebra of smooth functions on R^p, we can interpret $\rho(a_j)$ as test functions obtained by superposition of exponentials $\exp(ip_j x)$; by going to Fourier space, one recognises $[D, \rho(a_i)]$ as an insertion of $\not{p}_j = \gamma_\mu p_j^\mu$ at the vertex j (since $[\gamma^\mu \partial_\mu, \exp(ip_j x)] = i\not{p}_j$) and D^{-1} as the Dirac propagator. Tr denotes both a trace in the Clifford algebra and an integral over dx^p. Moreover, the usual conservation of impulsion at each vertex comes from the fact that products become convolution products in Fourier space and from the properties of the exponential. One gets moreover an overall conservation of momentum

$$\delta\left(p_0 + \sum_{i=1}^{n} p_i\right)$$

which shows that the result is a function of n independent variables. This type of diagram, with one insertion of an axial current, is quite standard in Quantum Field Theory. The p-summability of the Fredholm module is a translation of the fact that this type of diagram is convergent whenever the number of external legs is $\geq (p+1)$. Going back to x-space, we will give the 4-cocycle $\tau_0(f^0, f^1, f^2, f^3, f^4)$ when A is the algebra of smooth functions over an 4-dimensional spin manifold (F being the Dirac operator):

$$
\begin{aligned}
&\tau_0(f^0, f^1, f^2, f^3, f^4) = \\
&= \int f^0 \, df^1 \wedge df^2 \wedge df^3 \wedge df^4 + \\
&+ \frac{1}{12} \int f^0 f^1 f^2 f^3 f^4 R^{\mu}_{\nu\alpha\beta} R^{\nu}_{\mu\gamma\delta} \, dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta
\end{aligned}
\tag{64}
$$

where R is the Riemann curvature tensor. The general formula ([7]) for a manifold X of dimension n is:

$$
\begin{aligned}
\tau_0(f^0, \ldots, f^n) = \int f^0 \, df^1 \wedge \ldots \wedge df^n &+ (S^2 \omega_1)(f^0, \ldots, f^n) \\
&+ (S^4 \omega_2)(f^0, \ldots, f^n) + \ldots + \\
&+ (S^{n/2} \omega_{n/4}(f^0, \ldots, f^n)
\end{aligned}
\tag{65}
$$

where the ω_j are the differential forms $\omega_j = \hat{A}_j(p_1, \ldots, p_j)$ expressing the $\hat{A}$ genus of X, and S is the suspension operator of cyclic cohomology (sect. 5.1). Here, the ω_j are of degree $4j$ and should be viewed as a current $\tilde{\omega}_j$ of dimension $n - 4j$:

$$
\tilde{\omega}_j(f^0, \ldots, f^{n-4j}) = \int f^0 \, df^1 \wedge \ldots \wedge df^{n-4j} \wedge \omega_j .
\tag{66}
$$

A given Fredholm module (H, F) defines cyclic cocycles $\tau_n, \tau_{n+2}, \tau_{n+4} \ldots$ and therefore cyclic cohomology classes $[\tau_n], [\tau_{n+2}], [\tau_{n+4}] \ldots$. One can prove [7], that $S[\tau_n] = [\tau_{n+2}]$ where S is the operator introduced in §5.1. The number obtained from the pairing of e with $\tau_n \in H^n_\lambda$ was given in sect. 9.5. From this last property, it is clear that any of the characters $[\tau_n]$, for n large enough, defines an element of the even periodic cyclic group $H^{\text{eve}}_{\text{per}}$ (cf. sect. 5.3). This element is denoted $Ch^*(H, F)$ and called the character of the Fredholm module (H, F).

We do not explain how to construct odd-dimensional cyclic cocycles for an algebra A [7]. Roughly speaking, one has to build a Fredholm module for the algebra $A \otimes C_L$ where $C_L = \mathrm{C} + \mathrm{C}$ is the Clifford algebra generated by 1 and α with $\alpha^2 = 1$.

10.4. Noncommutative index theory ♠

We already know (sect. 9.5) that even cyclic cocycles pair with elements of $K_0(A)$, a generalization of the fact that one can obtain a number by integrating a characteristic class over the base of a vector bundle. Since a Fredholm module (H, F) over A (a generalization of an elliptic operator) provides us with a whole hierarchy of cyclic cocycles τ_n, it is clear that we can pair those cyclic cocycles against an arbitrary element $[e] \in K_0(A)$ ($[e]$ describing an equivalence class of finite projective modules over A and e being explicitly given as an idempotent in $M_k(A)$). The last result of section 10.3 shows actually that the interesting pairing is obtained between $[e] \in K_0(A)$ and $[\tau] = Ch^*(H, F)$ in $H^{\text{even}}_{\text{per}}$. The important result is that this number $\langle [e], [\tau] \rangle$ turns out to be the index of a Fredholm operator and, therefore, an integer. Writing

$$F = \begin{bmatrix} 0 & Q \\ P & 0 \end{bmatrix}$$

and

$$e = \begin{bmatrix} e^0 & 0 \\ 0 & e^1 \end{bmatrix},$$

one considers

$$eFe = \begin{bmatrix} 0 & Q_e \\ P_e & 0 \end{bmatrix}.$$

$P_e \doteq (eFe)^+$ is a Fredholm operator and its index turns out to be equal to $\langle [e], [\tau] \rangle$. The fact that this expression depends only and additively upon the class [e] is a remarkable property [7].

10.5. Noncommutative connections using Fredholm modules

Let ∇ be a noncommutative connection acting on a (right) finite projective module E over a unital $*$ algebra A, with values in the universal differential algebra $(\Omega(A), \delta)$ -cf.§9.4. Let also (H, F) be a (left) Fredholm module (with $F^2 = 1$). Then, as in §10.3, one builds the differential algebra

$$\Lambda = \oplus_q \Lambda^q$$

where Λ^q is the linear span of monomials $a_0 da_1 \dots da_q$ and where $da_q = i[F, a_q]$, $a_q \in A$. Here we no longer write explicitly the morphism ρ such that $\Lambda^0 = \rho(A)$. The universal map $(\Omega(A), \delta) \to (\Lambda, d)$ allows us to replace «abstract» Ω-connections by «concrete» Λ-connection. In the particular case $E = A$, such a connection will be described by an element

$$\omega = \sum_i a_i db_i = i \sum_i a_i [F, b_i]$$

of Λ^1 and its curvature Θ as

$$\Theta = d\omega + \omega^2 = i(F\omega + \omega F) + \omega^2,$$

indeed

$$d\omega = i[F, \omega]_g = i(F\omega + \omega F),$$

since it is a graded commutator. To make things even more concrete, we can take the example of a compact Riemannian manifold M of dimension n, with H the Hilbert space of L^2 spinors on M and where the operator F is the phase of the Dirac operator ($F = D|D|^{-1}$) ; *i.e.*, from the physical point of view, is the operator that distinguishes between positive and negative frequencies. For a four-dimensional manifold M , the above Fredholm module is p-summable for $p = 4 + \epsilon$, $\epsilon > 0$. On the other hand we mentioned in §10.3 the fact that $\Lambda^q \subset \mathcal{L}^{p/q}$ when the Fredholm module is p-summable . The curvature $\Theta \in \Lambda^2 \subset \mathcal{L}^{p/2}$, so that Θ is a Hilbert Schmidt operator ($\Theta \in \mathcal{L}^2$) whenever $p \leq 4$. This shows that the noncommutative Yang-Mills action (or, for that matter, the Maxwell action). $Trace(\Theta^*\Theta)$ is finite in dimension 4- ϵ. This is a nice reformulation of the corresponding fact in perturbative Quantum Field Theory (which have been known by more than forty years by physicists). One can prove (exactly as in Quantum Electrodynamics for example) that in dimension 4 the divergence of trace ($\Theta^*\Theta$) is only logarithmic ([41],[any book on QED]) and that its principal term (the coefficient of $\log(L)$ if L is a cutoff) can be identified with the classical action. Technically, the calculation of the principal term of such an operator P is given by Wodzicki residue or equivalently by the so-called Dixmier trace [41].

10.6. Θ-summable Fredholm modules ♠

In some circumstances, the algebra A in which we are interested is «so big» that one cannot find p-summable Fredholm modules. This happens in particular in Quantum Field Theory (the case of the algebra of quantum fields in the supersymmetric Wess-Zumino model is such an example and has been analysed in [29] [30]). However, it may be possible to find Fredholm modules (H, D, Γ) , with $\Gamma D = -D\Gamma$ as usual, but such that $\exp(-tD^2)$ is of trace class for any positive t . This is called a Θ-summable Fredholm module over A (the name coming from the analogy with Θ-functions) [25][26]. As in the finitely summable case, one can define even and odd Θ-summable modules. We will restrict our discussion here to the case of even modules. In order to explain how such a module defines an (even) entire cyclic cocycle $(\phi_{2n})_{n\in N}$ called the character of the module, the best is probably to give an analogy with the finitely summable case. In the p-summable situation, we saw how to construct a homomorphism π from the universal differential envelope ΩA to a concrete differential algebra Λ (replacing the symbol δa by the bounded operator $i[F,a]_g$ on the Hilbert space H) and how to build cyclic cocycles by using the natural closed graded trace on Λ . Here, in the case of Θ-summable Fredholm modules, the situation is similar, but more involved. One has to replace ΩA by the Zekri algebra $\epsilon A = QA \oplus QA$ and the differential algebra Λ by an algebra $\tilde{\mathcal{L}} = \mathcal{L} \oplus \mathcal{L}$ where $\mathcal{L}$ is a convolution algebra of operator-valued distributions $T(s)$ with support in R^+ and valued in bounded operators in H . The homomorphism π from ϵA to $\tilde{\mathcal{L}}$ is given by a homomorphism from A to $\tilde{\mathcal{L}}$ and an element of $\tilde{\mathcal{L}}$ of square 1. One can also define a natural trace τ on the algebra $\tilde{\mathcal{L}}$. The character of the module (H, D, Γ) is then defined as the trace on ϵA given by $\tau(\pi(x))$, *i.e.*, formally by the same formula that in the finitely summable case. The analysis required is however slightly involved and we refer to [25] for the details. The fact that the character is an entire cyclic cocycle is clear since it is a trace on ϵA , cf. sect. 7. For each n, one obtains in this way a ($2n+1$) linear form τ_{2n} on A , corresponding to the monomial $a^0\delta a^1 \dots \delta a^n$. By definition the components of the character are the members of the sequence $(\phi_{2n})_{n\in N}$ where

$$\phi_{2n} = \Gamma\left(n+\frac{1}{2}\right)\tau_{2n}.$$

The previous «abstract» definition of τ_{2n} allows to compute an explicit expression for τ_{2n} . Actually, there exist two different formulae for the character. The first one, given by [25] is the following.

$$\begin{aligned}\tau^{2n}(a^0,a^1,\dots,a^{2n}) = \mathrm{Tr}\int_{-\infty}^{+\infty} & F(im+\alpha)a^0[F(im+\alpha),a^1]\dots \\ & [F(im+\alpha),a^{2n}]e^{(im+\alpha)^2}\frac{dm}{\sqrt{\pi}}\end{aligned} \tag{69}$$

where $F(m) = (D+m\Gamma)(D^2+m^2)^{-1/2}$. The result is actually independent of $\alpha > 0$. Notice that,since $F^2(m) = 1$, if we *formally* permute trace and integration and set $\alpha = 0$ the right hand side reads simply

$$\int_{-\infty}^{+\infty}\mathrm{Tr}(F(m)a^0[F(m),a^1]\dots[F(m),a^{2n}])e^{-m^2}\frac{dm}{\sqrt{\pi}}.$$

The second formula was established in [29] and we describe it below. One first introduce a time variable t and set $x(t) = e^{-tH}xe^{+tH}$ where D may be, for example, the Dirac operator on a loop space and $H = D^2$ is a hamiltonian (laplacian). To avoid a possible confusion between the time derivative operator d/dt and the derivation which was called d in sect. 10.3, we change our notations here and denote the latter by Δ . Therefore $\Delta x = i[D,x]_g$, $\Delta^2 x = [x,D^2] = [x,H]$. It is then clear that $\Delta(x(t)) = i[D, x(t)]_g = e^{-tH}\Delta x e^{tH} = (\Delta x)(t)$. The relation between Δ and the time derivative is then the following:

$$\Delta^2 x(t) = [x(t), H] = \frac{dx}{dt}(t).$$

The formula for the character reads

$$\begin{aligned}
&\tau_\beta^{2n}(a^0, a^1, \ldots, a^{2n}) = \\
(71) \qquad &= (-\beta)^{-n} \int_{O<t_1<t_2 \cdots <t_{2n}<\beta} \mathrm{Tr}(\Gamma a^0(0)\Gamma[a^1(t_1), D]\Gamma[a^2(t_2), D] \ldots \\
&\Gamma[a^{2n}(t_{2n}), D] e^{-\beta H})\, dt_0 dt_1 \ldots dt_{2n}
\end{aligned}$$

where $a(t) = e^{-tH} a e^{+tH}$. The cohomology classes defined by τ_β^{2n} are independent of β. The link between these two expression has been studied in [28] but seems still unclear. The last formula is rather appealing from the physical point of view. It can be obtained formally from $a_0 da_1 \ldots da_n$ by replacing da_i by $\Delta a_i(t_i)$. The above formula can of course also be written by using a chronological T-product.

One can also prove [25] that evaluation of the character on an element e of $K_0(A)$ (represented as an idempotent of A) gives the index of the Fredholm operator $D_e^+ = (eDe)^+$.

11. KASPAROV KK-THEORY (GENERALITIES) ♠

The bivariant theory of Kasparov [38] (called KK-theory) is usually considered as a quite esoteric subject. However, it is somewhat at the root of most of what we discussed so far and we could have started the present review by discussing this theory. Our purpose here is only to give a glimpse of what the subject is, and we follow more or less the idea developed in [10] and [24]. The present section could actually be read just after section 2 where we show how to construct the universal differential envelope ΩA as well as the Cuntz algebra QA and the Zekri algebra ϵA from a given associative algebra A.

11.1. The group $KK^0(A, B)$

Let A and B be two (denumerably generated) algebras. Then one defines the abelian group $KK^0(A, B) = [qA, K \otimes B]$. For this definition to be meaningful, we have to give the definition of qA, of $K \otimes B$ and of the symbol [ ]. qA was already defined in section 2, it is the ideal of QA generated by the symbols $qa, a \in A$. $K \otimes B$ denotes the limit of

$$B \hookrightarrow \begin{pmatrix} B & O \\ O & O \end{pmatrix} \hookrightarrow \begin{pmatrix} M_2(B) & O \\ O & O \end{pmatrix} \hookrightarrow .$$

K stands for the algebra of compact operators. Finally $[D_1, D_2]$ denotes the space of homomorphisms from the algebra D_1 in the algebra D_2 quotiented by homotopy. Two such homomorphisms φ_1 and φ_2 are homotopic if one can find a homomorphism ψ from D_1 into $D_2[0,1] \doteq C([0,1]) \otimes D_2$ such that $\psi_{t=0} = \varphi_1$ and $\psi_{t=1} = \varphi_2$. The above definition is due to [10]. The original definition [38] of $KK^0(A,B)$ was to define it as the abelian group of homotopy classes of quasi-homomorphisms from A to B. A quasi-homomorphism being itself a pair (φ, φ') of homomorphisms from A into an algebra E such that for all $a \in A$, $\varphi(a) - \varphi'(a) \in K \otimes B$. The equivalence between the two definitions comes from the fact that the Cuntz algebra factorises pairs of homomorphisms (as we saw in sect. 2). We want to think of an element of $KK^0(A,B)$ as a generalized homomorphism from A to B.

11.2. The case of manifolds

Where A or B is the algebra $C(X)$ of continuous functions over the manifold X, one finds that $KK^0(\mathrm{C}, \mathrm{C}(X)) = K^0(X)$ is the group of K-theory of X introduced in sect. 9.3, and that $KK^0(\mathrm{C}(X), \mathrm{C}) = K_0(X)$ is the group of K-homology of X introduced in sect. 10.1. More generally, one obtains $KK^0(\mathrm{C}, A) = K_0(A)$ and $KK^0(A, \mathrm{C}) = K^0(A)$.

11.3. The Kasparov product for KK^0

Kasparov introduced a map

$$KK^0(A,B) \times KK^0(B,C) \to KK^0(A,C).$$

This map is bilinear (on both sides), associative and compatible with the bifunctorial properties of KK (indeed KK is a bifunctor between pairs of algebras and abelian groups). The original definition of the Kasparov product involving composition of pairs of homomorphisms is rather involved but in the present framework it just becomes a composition of homomorphisms (here, one has to prove that working with $q^2 A$ is equivalent to working with qA, a property which is not obvious). Notice that in the case of manifolds we get

$$KK^0(\mathrm{C}, C(X)) \times KK^0(C(X), \mathrm{C}) \to KK^0(\mathrm{CC})$$

but C can be considered as the algebra C (pt.) of functions over a point, so that

$$KK^0(\mathrm{CC}) = K^0(\mathrm{pt.}) = \mathrm{Z}$$

(since the dimension – an integer – is the only topological invariant of a complex vector space!). So we recover the (Z-valued) pairing between K-theory (vector bundles) and K-homology (elliptic operators).

11.4. The group $KK^1(A,B)$

The most direct definition is probably the following:

$$KK^1(A,B) = KK^0(SA,B) \qquad (\simeq KK^0(A,SB))$$

where

$$SA = \{f \in A[0,1]/f(0) = f(1) = 0\}$$

is the suspension of the algebra A.

Another definition makes use of the properties of algebras extensions: each element of

$$KK^1(A,B)$$

defines a half-split extension of B by A, *i.e.* an exact sequence

$$0 \to K \otimes B \to D \underset{\sigma}{\overset{\pi}{\rightleftarrows}} A \to 0$$

where the lift σ is a section but not necessarily an algebra homomorphism (hence the «half» of «half-split»).

A last definition similar to the Cuntz definition was proposed in [29]. Namely

$$KK^1(A,B) = [\epsilon A, K \otimes B].$$

ϵA being the Zekri algebra of A.

11.5. The Kasparov product for KK^1 and KK^0

Given these algebras A, B and C, one can define a product $\times$ with

$$KK^0(A,B) \times KK^0(B,C) \to KK^0(A,C),$$
$$KK^1(A,B) \times KK^0(B,C) \to KK^1(A,C),$$
$$KK^0(A,B) \times KK^1(B,C) \to KK^1(A,C),$$
$$KK^1(A,B) \times KK^1(B,C) \to KK^0(A,C),$$

of course one could be tempted of defining $KK^{n+1}(A,B) = KK^n(SA,B)$ for any $n > 1$ but it can be proved (Bott periodicity in KK-theory) that $KK^{n+2}(A,B)$ is isomorphic with $KK^n(A,B)$ so that only KK^0 and KK^1 are relevant.

12. FRONTIERS

In the previous chapters, it has not been possible to cover all possible aspects of this fastly developing field. In the present section we want to indicate a few other topics along with some references.

– Yang-Mills equations on noncommutative spaces. The moduli space for connections minimizing the Yang-Mills functional on a noncommutative two torus (the C^*-algebra generated by two unitary operators U_1 and U_2, subject to the condition $U_1 U_2 = \lambda U_2 U_1$) has been analysed in [39]. It was shown that it is a commutative torus T^2.

– Noncommutative Riemannian structures. The key observation [26] is that if M is a compact, spin, Riemannian manifold and D is the Dirac operator acting on the L^2 sections of the bundle of spinors, it is possible to reconstruct the geodesic distance $d(x, y)$ between two points from the formula

$$d(x,y) = \sup\{|f(y) - f(x)|; \quad f \in C(M), \quad \| D, f \| \leq 1\}.$$

The theory can be generalized by replacing the preceding data by an arbitrary unbounded Fredholm module over a noncommutative algebra A.

– Dirac operators coupled to connections in noncommutative spaces. If (h, D) is a «Connes module» for the algebra A (as in sections 10.2, 10.3), then

$$F = \begin{bmatrix} 0 & D \\ D^{-1} & 0 \end{bmatrix}$$

defined on $H = h + h$ is also a left module on A. Moreover, one can consider a Λ-connection ∇ on a right finite projective module E (as in sect. 9.4 and 10.5), (Λ, δ) being a graded differential algebra with $\Lambda^0 = A$. One can then build the space

$$E_H = E \otimes_A H$$

on which the operator $\rho = \nabla \otimes iF$ acts. Notice that ρ is not a connection in the general sense but $\rho^2 = 1 - \Theta$ where Θ is the curvature of ∇. If we think of D as a Dirac operator and ∇ as a usual connection, we see that ρ generalizes the idea a Dirac operator coupled to a gauge potential [40].

– Group actions and crossed products. Let G be a group acting by automorphisms in the algebra A $(g \in G \to \alpha(g) \in \mathrm{Aut}\, A)$. Then we build $G \times_\alpha A$ as the space of equivalence classes $(g, a) \sim (gk, \alpha(k^{-1})a)$. One can also represent an element λ of $G \times_\alpha A$ as a map from G to A equivariant under α. This crossed product is endowed with an algebra structure under the convolution product

$$(\lambda.\mu)(k) = \sum_{h \in G} \lambda(h)\alpha(h)[\mu(h^{-1}k)].$$

When A is the commutative algebra $C(X)$ in the particular cases where X/G is a «bad» quotient (and where the standard tools of usual geometry break down), the noncommutative geometry of these crossed products offers a good alternative to the «usual» geometry of X/G.

Remarks

It should be clear that what we discussed so far are topics belonging to «Noncommutative differential geometry». Indeed, in the study of commutative algebras (*i.e.*, in «usual» geometry), one usually discusses measure theory *before* topology, topology *before* differential geometry and Lie groups (for example) *after* differential geometry. The same logical path can also be followed in the noncommutative case.

On one side of noncommutative geometry, we have non commutative measure theory. Usual measure theory tells us how to get a number $\varphi(f)$ out of a function f (belonging to a commutative von Neumann algebra (8) L of essentially bounded measurable functions on a space X) via the relation $\varphi(f) = \int_X f \, d\mu$ where μ is a measure on X ; φ is called a weight on L. The main point is that any pair (L, φ) of a commutative von Neumann algebra L and weight φ can be obtained as above from a space X with a measure μ. Therefore the classification of pairs (L, φ), with L commutative, amounts to a classification of measured spaces. It is therefore clear that the theory of weights on noncommutative von Neumann algebras can be called a noncommutative measure theory. To make the link with physics, remember that to each weight φ on L corresponds a one parameter group σ_t of automorphisms of L (σ_t is the identity whenever L is commutative) and that, in the case where L is a matrix algebra $M_m(\mathbb{C})$ and φ is given by $\varphi(f) = \mathrm{Trace}(fe^{-\beta H})/\mathrm{Trace}(e^{-\beta H})$, then σ_t describes the evolution of the system: $\sigma_t(f) = e^{itH} f e^{-itH}$. We refer to [42] for a survey of noncommutative measure theory.

On the «other side» of noncommutative differential geometry we find the so-called Quantum Groups (cf. [43],[44]). These are not groups but are noncommutative algebras A on which has been defined a co-product. The group law on a group G is a map $G \times G \to G$ endowed with some well-known properties. This can be translated in a co-product $A \to^{\Delta} A \otimes A$ where A is the (commutative) algebra of complex functions on G (if $f \in a$ then $\Delta f(g_1, g_2) = f(g_1 g_2)$); these co-products have particular properties obtained by «dualizing» the axioms of a group. Quantum Groups are then obtained by replacing the commutative algebra A by an arbitrary noncommutative algebra. Quantum Groups (and their representations) seem to play a role in physics (particularly in relation with two-dimensional statistical models).

(8) Elements of the algebra L can be also considered as operators in the Hilbert space $H = L^2(X, \mu)$ where they act by multiplication. It is easy to see that L is equal to its commutant L' in H and therefore also to its bicommutant L''. Hence L is a von Neumann algebra.

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